

A Large N Duality via a Geometric Transition

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We propose a large N dual of 4d, $\mathcal{N} = 1$ supersymmetric, $SU(N)$ Yang-Mills with adjoint field Φ and arbitrary superpotential $W(\Phi)$. The field theory is geometrically engineered via D-branes partially wrapped over certain cycles of a non-trivial Calabi-Yau geometry. The large N , or low-energy, dual arises from a geometric transition of the Calabi-Yau, where the branes have disappeared and have been replaced by suitable fluxes. This duality yields highly non-trivial exact results for the gauge theory. The predictions indeed agree with expected results in cases where it is possible to use standard techniques for analyzing the strongly coupled, supersymmetric gauge theories. Moreover, the proposed large N dual provides a simpler and more unified approach for obtaining exact results for this class of supersymmetric gauge theories.

1. Introduction

Partially wrapping D-branes over non-trivial cycles of non-compact geometries yields large classes of interesting gauge theories, depending on the choice of geometry. It has also been suggested in [1,2] that $N \gg 1$ D-branes, wrapped over cycles, have a dual description (in a suitable regime of parameters) involving transitions in geometry, where the D-branes have disappeared and have been replaced by fluxes. This duality can be reformulated and explained as a geometric flop in the context of M-theory propagating on G_2 holonomy manifolds [3,4]. In this paper, we use these ideas to propose a new class of dualities.

The simplest case, which will be the main focus of this paper, corresponds to an $\mathcal{N} = 1$ supersymmetric gauge theory with adjoint chiral superfield Φ and tree-level superpotential

$$W_{tree} = \sum_{p=1}^{n+1} \frac{g_p}{p} \text{Tr } \Phi^p \equiv \sum_{p=1}^{n+1} g_p u_p, \quad (1.1)$$

where the gauge group can be either $SU(N)$ or $U(N)$, depending on whether or not we treat g_1 as a Lagrange multiplier imposing tracelessness of Φ . For simplicity, we generally refer to $U(N)$, with the understanding that the $SU(N)$ can be obtained by imposing the Lagrange multiplier condition. Without the superpotential (1.1), the theory would be $\mathcal{N} = 2$ super-Yang-Mills. The theory with superpotential (1.1) arises [5] by wrapping N type IIB D5 branes on special cycles of certain Calabi-Yau geometries; the choice of n and the parameters g_p are given by the geometry. Using the corresponding geometric transition, we construct a dual theory without the D-branes, but with suitable fluxes. There is also a mirror IIA description, involving D6 branes wrapping 3 cycles. The IIB description is simpler, in that there are no worldsheet instanton corrections to the superpotential. However, the IIA perspective is useful for *explaining* the origin of these dualities, as they are related to geometric flop transitions in M-theory on G_2 holonomy geometries [3].

The classical theory with superpotential (1.1) has many vacua, where the eigenvalues of Φ are various roots a_i of

$$W'(x) = \sum_{p=0}^n g_{p+1} x^p \equiv g_{n+1} \prod_{i=1}^n (x - a_i). \quad (1.2)$$

In the vacuum where classically $P(x) \equiv \det(x - \Phi) = \prod_{i=1}^n (x - a_i)^{N_i}$, the gauge group is broken as

$$U(N) \rightarrow \prod_{i=1}^n U(N_i) \quad \text{with} \quad \sum_{i=1}^n N_i = N. \quad (1.3)$$

In the geometric construction [5], this is seen because we can wrap N_i D5 branes on any of n choices of $S^2 \cong \mathbf{P}^1$. Such a vacuum exists for any partition of $N = \sum_{i=1}^n N_i$.

Applying the proposal of [1,2] to each S^2 , a transition occurs where we are instead left with n S^3 s. As we discuss, the non-compact Calabi-Yau geometry is now given by the following surface in \mathbf{C}^4 :

$$W'(x)^2 + f_{n-1}(x) + y^2 + z^2 + v^2 = 0, \quad (1.4)$$

with $W'(x)$ the degree n polynomial (1.2) and $f_{n-1}(x)$ a degree $n-1$ polynomial. As for any Calabi-Yau, we can form an integral basis of 3-cycles, A_i and B_i , which form a symplectic pairing

$$(A_i, B_j) = -(B_j, A_i) = \delta_{ij}, \quad (A_i, A_j) = (B_i, B_j) = 0, \quad (1.5)$$

with the periods of the Calabi-Yau given by the integral of the holomorphic 3-form Ω over these cycles. In the present case (1.4), we have $i = 1 \dots n$, with the A_i cycles compact and the B_i non-compact. We denote the periods as

$$\int_{A_i} \Omega \equiv S_i, \quad \int_{B_i}^{\Lambda_0} \Omega \equiv \Pi_i = \frac{\partial \mathcal{F}}{\partial S_i} \quad (1.6)$$

with $\mathcal{F}(S_i)$ the prepotential. Λ_0 is a cutoff needed to regulate the divergent B_i integrals; this is actually an infrared cutoff in the geometry integral, which will naturally be identified with the ultraviolet cutoff of the 4d QFT. Using (1.6), the polynomial $f_{n-1}(x)$ in (1.4) is to be solved for in terms of the n A_i periods S_i .

As in [2], the dual theory obtained after the transitions to the geometry (1.4) has a superpotential due to fluxes through the 3-cycles of (1.4):

$$-\frac{1}{2\pi i} W_{eff} = \sum_{i=1}^n (N_i \Pi_i + \alpha_i S_i), \quad (1.7)$$

with N_i 3-form $(H_R + \tau H_{NS})$ flux through A_i and α_i 3-form flux $(H_R + \tau H_{NS})$ through B_i [6,7]. If not for the superpotential (1.7), the dual theory would yield a 4d, $\mathcal{N} = 2$ supersymmetric, $U(1)^n$ gauge theory, with the S_i the $\mathcal{N} = 1$ chiral superfields in the $\mathcal{N} = 2$ $U(1)^n$ vector multiplets. In terms of this field theory, the superpotential (1.7) corresponds to breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$ by adding electric and magnetic Fayet-Iliopoulos terms [8]. There will be $\mathcal{N} = 1$ supersymmetric vacua, with the S_i massive and thus fixed to some

particular $\langle S_i \rangle$, but with the $\mathcal{N} = 1$ $U(1)^n$ gauge fields left massless. In the applications we consider, all $\alpha_j \sim 1/g_0^2$, the bare gauge coupling of the gauge theory; this combines in a natural way with Λ_0 , replacing the cutoff with the physical scale Λ of the gauge theory.

The duality proposal, generalizing that of [2], is that these $U(1)^n$ gauge fields coincide with those of the original theory (1.3) after the $SU(N_i)$ get a mass gap and confine. In particular, the *exact* quantum effective gauge couplings $\tau_{ij}(g_r, \Lambda; N_i)$ of the remaining massless $U(1)^n$ gauge fields should be given by the prepotential of the above dual, $\tau_{ij} = \partial^2 \mathcal{F} / \partial S_i \partial S_j$, evaluated at $\langle S_i \rangle$. Further, as in [2], the S_j are to be identified with the $SU(N_j)$ “glueball” chiral superfields $S_j = -\frac{1}{32\pi^2} \text{Tr}_{SU(N_j)} W_\alpha W^\alpha$, whose first component is the $SU(N_j)$ gaugino bilinear. Finally, we claim that the superpotential (1.7) is the *exact* quantum effective superpotential of the low-energy $SU(N)$ theory with superpotential (1.1), in the vacuum with the Higgsing (1.3).

Note that the $U(1)^n$ dual theory only knows about the values of the N_i via the coefficients appearing in (1.7). In particular, the $\Pi_i(S_j; g_r, \Lambda)$ and $\mathcal{F}(S_i; g_r, \Lambda)$ are completely *independent* of the N_i , depending only on Λ and the parameters g_r via (1.4). Upon adding (1.7) to the dual theory, one obtains $\langle S_i \rangle$ which are complicated functions of the N_i , g_r , and Λ . Integrating out the S_i gives the exact quantum 1PI effective superpotential $W_{eff}(g_r, \Lambda, N_i)$ of the original theory.

The geometric transition leads to a new duality, which can be stated in purely field theory terms: the $U(N)$ theory with adjoint and superpotential (1.1) is dual to a $U(1)^n$ theory and superpotential (1.7). This duality is reminiscent of that of [9].

The above duality makes some highly non-trivial predictions for the exact $U(1)^n$ gauge couplings $\tau_{ij}(g_r, \Lambda)$ and the exact effective superpotential $W_{eff}(g_r, \Lambda)$. This allows us to check the duality, by comparing with the exact results which can (at least in principle) be obtained for these quantities via a direct field theory analysis. The above quantities can be exactly obtained (again, at least in principle) by viewing the $\mathcal{N} = 1$ $U(N)$ theory with adjoint Φ and superpotential (1.1) as a deformation of $\mathcal{N} = 2$, and using the known exact results for $\mathcal{N} = 2$ field theories. We find perfect agreement between these results, which is a highly non-trivial check of our proposed duality.

The organization of this paper is as follows: In section 2 we review the large N duality of [2] for $\mathcal{N} = 1$ Yang-Mills theory, and briefly discuss the extension to include massive flavors in the fundamental of $U(N)$. In section 3, we discuss how to geometrically engineer the general $\mathcal{N} = 1$ theory with adjoint and superpotential (1.1). In section 4 we propose the large N dual of these theories via the transition in the CY geometry where

S^2 s are blown down, S^3 s are blown up, and the branes have been replaced with fluxes. In section 5 we analyze the $U(N)$ theory with adjoint and superpotential using standard supersymmetric field theory tools. In section 6 we specialize these results to the case of the cubic superpotential. In section 7 we analyze the proposed large N duals and show how the leading order computation of gauge theory based on gaugino condensate follows from monodromies of the geometry. In section 8 we specialize to the cubic superpotential and compute exact results for the quantum corrected superpotential using the proposed dual. We find perfect agreement with the results based on a direct gauge theory analysis. In appendix A we present the details of the analysis for one of the field theory examples, and in appendix B we discuss the series expansion for computing the periods for the case of cubic superpotential.

2. Review of the large N duality for $\mathcal{N} = 1$ Yang-Mills

Consider type IIA string theory on a non-compact Calabi-Yau threefold of T^*S^3 , i.e. the conifold, with defining equation given by

$$x^2 + y^2 + z^2 + v^2 = \mu,$$

and consider wrapping N D6 branes on the S^3 , with the unwrapped dimensions filling the Minkowski spacetime. This gives rise to a 4d $\mathcal{N} = 1$ $U(N)$ pure Yang-Mills theory. The duality proposed in [2], which was motivated by embedding the large N topological string duality of [1] into superstrings, states that in the large N limit this theory is equivalent to type IIA strings propagating on the blow up of the conifold. This is a geometry involving a rigid sphere \mathbf{P}^1 , where the normal bundle to the \mathbf{P}^1 in the CY is given by a $\mathcal{O}(-1) + \mathcal{O}(-1)$ bundle over it (i.e. two copies of the spinor bundle over the sphere). The branes have disappeared and have been replaced by an RR flux through \mathbf{P}^1 and an NS flux on the dual four cycle [2]. This duality has been embedded into M-theory, where it admits a purely geometric interpretation [3,4]. The $SU(N)$ gauge theory decouples from the bulk in the limit where the size S of the blowup sphere \mathbf{P}^1 is small. The size S is fixed in terms of the units of flux, and the appropriate decoupling limit is large N . S gets identified [2] with the glueball superfield $S = -\frac{1}{32\pi^2} \text{Tr} W_\alpha W^\alpha$ of the $SU(N)$ theory, so its expectation value corresponds to gaugino condensation in the $SU(N)$ theory.

As noted in [2] one can also consider the mirror description of this geometry, which is simpler to work with (as the worldsheet instanton corrections to spacetime superpotential

are absent). This corresponds to switching from IIA to IIB theory and reversing the arrow of transition: the original $U(N)$ theory is obtained from type IIB $D5$ branes wrapped around the \mathbf{P}^1 in the blown up conifold geometry and, in the large N limit, this is equivalent to type IIB on the deformed conifold background:

$$f = x^2 + y^2 + z^2 + v^2 - \mu = 0.$$

The deformation parameter μ will, again, be identified with the $SU(N)$ glueball superfield. Rather than the N original $D5$ branes, there are now N units of RR flux through S^3 , and also some NS flux through the non-compact cycle dual to S^3 . This mirror description is related to a particular limit of the large N duality proposed in [10] and [11].

The value of the modulus μ is fixed [2] by the fluxes, and this is captured by a superpotential for S , whose first component is proportional to μ . Specializing (1.5) and (1.6) to the conifold, we have a single compact 3-cycle $A \cong S^3$, and a single dual, non-compact 3-cycle B . The A period of the holomorphic 3-form Ω is S . There are N units of RR flux through A , and the NS flux α through B ; α is identified with the bare coupling of the 4d $U(N)$ gauge theory.

The holomorphic three-form Ω is given by

$$\Omega = \frac{dx dy dz dv}{df} \sim \frac{dx dy dz}{v} = \frac{dx dy dz}{\sqrt{\mu - x^2 - y^2 - z^2}}$$

The 3-cycles A and B can be viewed as 2-spheres spanned by a real subspace of y, z fibered over x , as in [12,13,14], and integrating Ω over the fiber y, z yields a one-form ω in the x -plane:

$$\int_{S^2} \Omega \sim dx \sqrt{x^2 - \mu} = \omega.$$

The A-cycle, projected to the x -plane, becomes an interval between $x = \pm\sqrt{\mu}$. Thus the A-period is given by:

$$S = \int_A \Omega = \frac{1}{2\pi i} \int_{-\sqrt{\mu}}^{\sqrt{\mu}} dx \sqrt{x^2 - \mu} = \frac{\mu}{4}$$

The B-period can be viewed as an integral from $x = \sqrt{\mu}$ to infinity. However this integral is divergent, and thus must be cutoff to regulate the infinity. Giving S dimension 3, x has dimension $3/2$, so we put the cutoff at $x = \Lambda_0^{3/2}$ where Λ_0 has mass dimension 1:

$$\Pi = \frac{1}{2\pi i} \int_{\sqrt{\mu}}^{\Lambda_0^{3/2}} dx \sqrt{x^2 - \mu} = \frac{1}{2\pi i} \left(\frac{1}{2} \Lambda_0^3 - 3S \log \Lambda_0 - S(1 - \log S) \right) + \mathcal{O}(1/\Lambda_0)$$

Note that, under $\Lambda_0^3 \rightarrow e^{2\pi i} \Lambda_0^3$, $\Pi \rightarrow \Pi - S$, shifting the B period by an A period. Using the fact that we have N units of RR flux through S^3 and α units of NS flux through the B-cycle, we find the superpotential [2]:

$$W_{eff} = N[3S \log \Lambda_0 + S(1 - \log S)] - 2\pi i \alpha S.$$

α is related to the bare coupling constant of the $SU(N)$ gauge theory by $2\pi i \alpha = 8\pi^2/g_0^2$. The coefficient of S in the above superpotential is given by

$$S(3N \log \Lambda_0 - 2\pi i \alpha),$$

which is the geometric analog of the running of the coupling. α depends on Λ_0 in such a way that the above quantity is finite as $\Lambda_0 \rightarrow \infty$:

$$\frac{8\pi^2}{g^2(\Lambda_0)} = \text{const.} + 3N \log \Lambda_0,$$

which is exactly the expected running of the coupling constant for the 4d $\mathcal{N} = 1$ $U(N)$ Yang-Mills theory. The upshot is to replace the cutoff Λ_0 in the above expression with the scale of the gauge theory, which we will denote by Λ . We thus have for the superpotential

$$W_{eff} = S \log[\Lambda^{3N}/S^N] + NS$$

(the linear term NS is a matter of convention and defines what one means by the physical scale Λ). This is indeed the superpotential of [15] for the massive glueball S . Integrating out S via $dW_{eff}/dS = 0$ leads to the N supersymmetric vacua of $SU(N)$ $\mathcal{N} = 1$ supersymmetric Yang-Mills:

$$\langle S \rangle = e^{2\pi i k/N} \Lambda^3, \quad k = 1, \dots, N.$$

2.1. Gauge Theoretic Reformulation of the duality

We can formulate the above large N duality in purely gauge theoretic terms. The conifold geometry without the fluxes corresponds to an $\mathcal{N} = 2$ $U(1)$ gauge theory with a charged hypermultiplet [16]. Turning on fluxes is equivalent to adding electric and magnetic Fayet-Iliopoulos superpotential terms, which softly break $\mathcal{N} = 2$ to $\mathcal{N} = 1$. The $\mathcal{N} = 2$ vector multiplet consists of a neutral $\mathcal{N} = 1$ chiral superfield S and an $\mathcal{N} = 1$ photon. The $\mathcal{N} = 1$ $U(1)$ photon is left massless, and is to be identified with the overall $U(1) \subset U(N)$

of the original $\mathcal{N} = 1$ theory. The $\mathcal{N} = 1$ chiral superfield S gets a mass, and is to be identified with the massive glueball chiral superfield S of the $SU(N)$ theory.

The identification of the $U(1)$ of the dual theory with the $U(1) \subset U(N)$ is consistent with the fact that minimization of the superpotential gives rise to

$$N \frac{\partial \Pi}{\partial S} + \alpha = N\tau + \alpha = 0$$

where we used the special geometry to connect the periods of the B-cycles with the coupling constant τ of the $U(1)$. Note that the coupling of the $U(1)$ theory is $-\alpha/N$ as it should be where $-\alpha$ is the bare coupling of the $U(N)$ theory and $U(1)$ is identified with $1/N$ times the identity matrix in $U(N)$ adjoint. In fact the “charged hypermultiplet” of the $U(1)$ is nothing but the baryon field of the original $U(N)$ theory. To see this note that before turning on the RR flux on S^3 , wrapping a $D3$ brane around it gives a charged hypermultiplet. Turning on the RR flux, induces N units of fundamental charge on it, as noted in the context of AdS/CFT correspondence in [17,18,19]. After turning on the flux the field is not allowed by itself, i.e., it is attached to N fundamental strings going off to infinity. Thus after the FI deformations of the superpotential it is slightly misleading to think of the $U(1)$ theory as having a fundamental hypermultiplet. In that context one can simply view this as an effective $U(1)$ theory with the SW $\mathcal{N} = 2$ geometry as would have been the case with a fundamental hypermultiplet.

2.2. Adding Massive Fields

As discussed in [2], we can also consider adding some quark chiral superfields, in the fundamental representation of $SU(N)$. In the type IIB description this is done by taking a D5 brane wrapping a holomorphic 2-cycle not intersecting the \mathbf{P}^1 , but separated by a distance ρ , where ρ is proportional to the mass of the hypermultiplet, as the matter comes from strings stretching between the non-compact brane and the N branes wrapped on \mathbf{P}^1 . If (ζ_1, ζ_2) denote the $\mathcal{O}(-1) + \mathcal{O}(-1)$ bundle over \mathbf{P}^1 , the 2-cycle is the curve $(\zeta_1, \zeta_2) = (\rho, 0)$ over a point on \mathbf{P}^1 . Passing this through the conifold transition, which in these coordinates is given by

$$\zeta_1 a - \zeta_2 b = \mu,$$

and rewriting it by a change of variables in the form

$$F(x, y) = x^2 + y^2 - \mu = \zeta_2 b,$$

we have a D5 brane wrapping a 2-cycle given by $\zeta_2 = 0$ and $x = \rho$. Since here x has dimension $3/2$, and ρ should be proportional to the mass m_0 , we identify $\rho = m_0 \Lambda_0^{\frac{1}{2}}$. As discussed in [20] such a D-brane gives rise to an additional spacetime superpotential

$$\Delta W_{eff} = \frac{1}{2} \int_{m_0 \Lambda_0^{1/2}}^{\Lambda_0^{3/2}} dx \sqrt{x^2 - \mu} = S \log \left(\frac{m_0}{\Lambda_0} \right) + \mathcal{O}\left(\frac{1}{\Lambda_0}\right).$$

This gives the running of the mass parameter with the cutoff Λ_0 . We define the renormalized mass by $m/\Lambda = m_0/\Lambda_0$. Generalizing to any number of matter fields in the fundamental representation, with mass matrix m , we find

$$\begin{aligned} W_{eff} &= S \log[\Lambda^{3N}/S^N] + NS + S \text{Tr} \log[m/\Lambda] \\ &= S \left[\log \frac{\Lambda^{3N-N_f} \det m}{S^N} + N \right]. \end{aligned}$$

Integrating out S via $dW_{eff}/dS = 0$ yields the correct field theory result:

$$S^N = \Lambda^{3N-N_f} \det m.$$

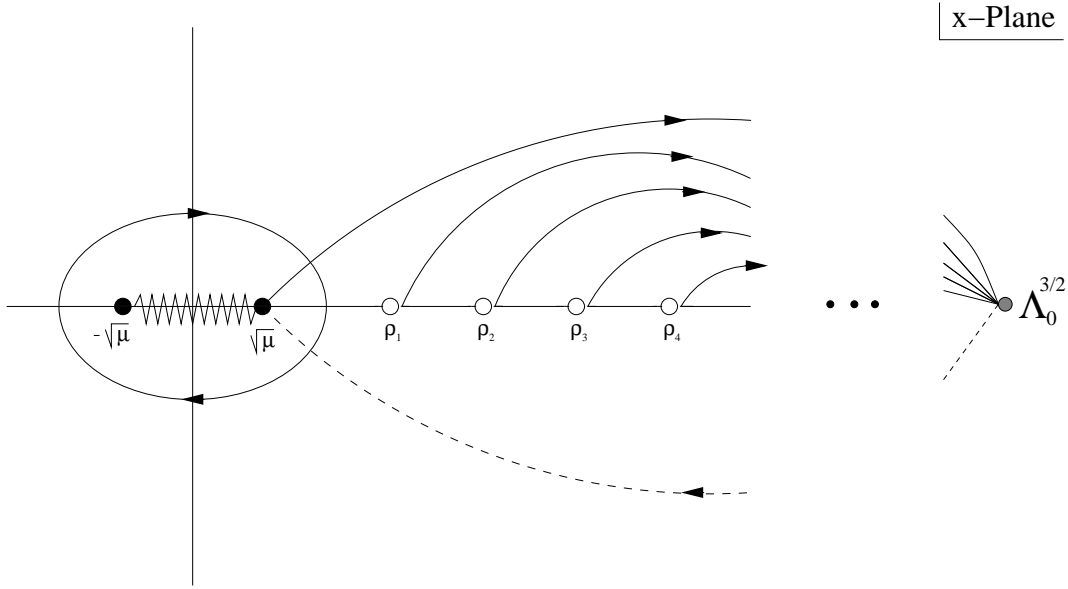


Figure 1: Location of the branch cut in the x -plane. Contours of integration of the different periods of the geometry including those coming from massive fields.

3. Geometric engineering $\mathcal{N}=1$ theories with adjoint Φ and superpotential $W_{tree}(\Phi)$

The $\mathcal{N} = 1$ $SU(N)$ Yang-Mills theory of the previous section can be regarded as a special case of the more general theory with adjoint Φ and superpotential as in (1.1),

$$W_{tree}(\Phi) = \sum_{p=1}^{n+1} \frac{g_p}{p} \text{Tr} \Phi^p. \quad (3.1)$$

For $n = 1$, the adjoint gets a mass $m = g_2$ and we recover the case reviewed in the previous section. We here review the geometric construction of [5] for general n .

For $W_{tree}(\Phi) = 0$, the 4d field theory would be pure $\mathcal{N} = 2$ Yang-Mills system. To geometrically engineer that, all we need is a \mathbf{P}^1 in a Calabi-Yau manifold for which the normal bundle is $\mathcal{O}(-2) + \mathcal{O}(0)$ (i.e. it has the same normal geometry as if the \mathbf{P}^1 were in a $K3$). If we wrap N D5 branes around the \mathbf{P}^1 we obtain an $\mathcal{N} = 2$ $U(N)$ gauge theory in the uncompactified worldvolume of the D5 brane. The adjoint scalar Φ gets identified with the deformations of the brane in the $\mathcal{O}(0)$ direction, normal to the \mathbf{P}^1 .

To describe the geometry in more detail, let z denote the coordinate in the north patch of \mathbf{P}^1 and $z' = 1/z$ in the south patch. Let x, x' denote the coordinate of $\mathcal{O}(0)$ direction in the north and south patches respectively, and let u, u' denote the coordinates of $\mathcal{O}(-2)$ in the north and south patches respectively. Then we have

$$z' = 1/z, \quad x' = x, \quad u' = uz^2. \quad (3.2)$$

There is a continuous family of \mathbf{P}^1 s, labeled by arbitrary x , at $u = 0 = u'$. Each of the N D5 branes can wrap a \mathbf{P}^1 at any value of x . In the $\mathcal{N} = 2$ gauge theory living in the unwrapped directions, this freedom to choose any x for each brane corresponds to moving along the Coulomb branch, with the a_i of each brane corresponding to an eigenvalue of the adjoint field Φ .

This connection between x and the Coulomb branch moduli makes it clear how the geometry must be deformed to obtain the $\mathcal{N} = 1$ theory with superpotential (3.1). Rather than having the \mathbf{P}^1 , with coordinate z and z' at the point $u = u' = 0$, for arbitrary x , it should exist only for particular values of x , namely the values $x = a_i$ where $W'(x) \equiv g_n \prod_{i=1}^n (x - a_i) = 0$. This is the case if (3.2) is deformed to

$$z' = 1/z, \quad x' = x, \quad u' = uz^2 + W'(x)z, \quad (3.3)$$

which is indeed only compatible with $u = u' = 0$ at the n choices of $x = a_i$ where $W'(x) = 0$. Note that now we can distribute the N D5-branes among the vacua a_i , i.e. N_i branes wrapping the corresponding S^2 at $x = a_i$. This gives a geometric realization of the breaking of $U(N) \rightarrow \prod_i U(N_i)$.

4. Large N Duality Proposal

We now obtain the large N dual of the $U(N)$ theory with adjoint Φ and superpotential $W_{tree}(\Phi)$ by considering the geometric transition where each of the n \mathbf{P}^1 's have shrunk and have been replaced by a finite size S^3 . As already mentioned, the sizes of the n S^3 s will correspond to the non-zero gaugino condensation expectation values in the n factors of $\mathcal{N} = 1$ non-Abelian gauge groups in (1.3). The needed blow-down of the n \mathbf{P}^1 s of the geometry of the previous section has been discussed in [21] and we will review it here. We start with the defining equation (3.3). Its blowdown can be obtained by the change of variables as follows: define $x_1 \equiv x$, $x_2 \equiv u'$, $x_3 \equiv z'u'$, $x_4 \equiv u$; using (3.3), these satisfy

$$x_2 x_4 - x_3^2 + x_3 W'(x_1) = 0.$$

By completing the square involving x_3 and W' and redefining the variables slightly we obtain the equation

$$W'(x)^2 + y^2 + z^2 + v^2 = 0. \quad (4.1)$$

This geometry is singular, even for a generic $W'(x)$; near each critical point of $W(x)$ it has the standard conifold singularity. The large N dual follows from desingularizing the geometry (4.1), allowing the n S^3 s to have finite size, rather than zero size as in (4.1).

4.1. Desingularization of the Geometry

Consider the most general desingularization of (4.1), subject to the restriction of [13] that the deformation be a normalizable mode. For the case at hand, as W'^2 is a polynomial of degree $2n$, the most general desingularization of (4.1) subject to the normalizability restriction is to add a polynomial $f_{n-1}(x)$ of degree $n-1$ in x [14], giving the geometry

$$W'(x)^2 + f_{n-1}(x) + y^2 + z^2 + v^2 = 0. \quad (4.2)$$

Under this deformation, each of the n critical points $x = a_i$ (1.2) (where $W' = 0$) splits into two, which we denote as a_i^+ and a_i^- .

As in the case of the conifold, the period integrals of the holomorphic three-form over the A_i and B_i cycles can be written as integrals of an effective one-form ω over projections of the cycles to the x plane. As in the conifold case, the non-trivial 3-cycles have simple projections to the x plane. The one-form ω is given by doing the Ω integral over the fiber S^2 cycles (corresponding to the y, z, v coordinates on the surface (3.3)); this gives

$$\omega = dx \sqrt{W'^2(x) + f_{n-1}(x)}. \quad (4.3)$$

Therefore, the periods of the holomorphic three-form Ω over the n 3-cycles A_i of (4.2), which are compact 3-spheres, are given by,

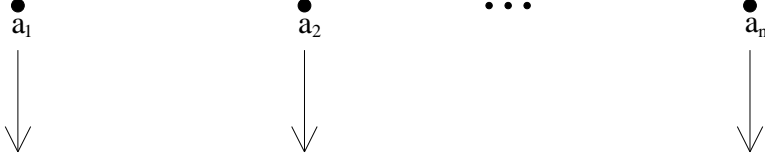
$$S_i = \pm \frac{1}{2\pi i} \int_{a_i^-}^{a_i^+} \omega \quad (4.4)$$

where the sign depends on the orientation; the periods over the dual B_i cycles are

$$\Pi_i = \frac{1}{2\pi i} \int_{a_i^+}^{\Lambda_0} \omega. \quad (4.5)$$

The map between the n coefficients in $f_{n-1}(x)$ and the S_i can thus be obtained by direct computation, and $f_{n-1}(x)$ can then be solved for as particular functions $f_{n-1}(x; S_i)$.

Singular Geometry



Non-singular Geometry

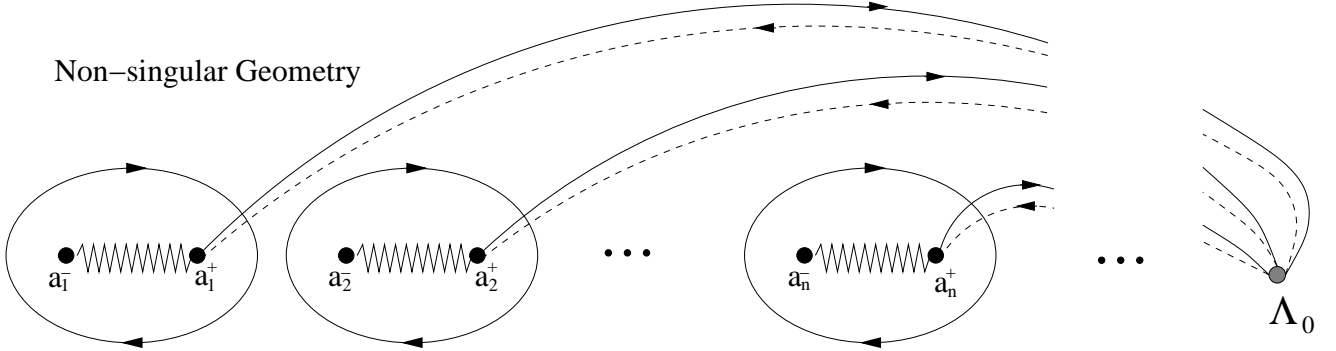


Figure 2: Geometry before and after introducing the deformation $f_{n-1}(x)$. The choice of branch cuts and integration contours for the different periods is also shown. Dashed lines are paths on the lower sheet.

As we already mentioned, the n values of S_i are mapped under the duality to the n glueball fields $S_i = -\frac{1}{32\pi^2} \text{Tr}_{SU(N_i)} W_\alpha W^\alpha$ for the non-Abelian factors in (1.3). (The S_i can be defined in a gauge invariant way.) Just as with the case of pure $\mathcal{N} = 1$ $U(N)$ Yang-Mills, the S_i of the dual theory will become massive and obtain particular expectation values thanks to a superpotential W_{eff} , with the expectation values $\langle S_i \rangle$ determined from finding the critical points of W_{eff} . The dual superpotential W_{eff} arises from the non-zero fluxes left after the transition.

Rather than having D-branes, as present before the transition, the above deformed geometry will have N_i units of H_R flux through the i -th S^3 cycle A_i . In addition, there is an H_{NS} flux α through each of the dual non-compact B_i cycles, with $2\pi i\alpha = 8\pi^2/g_0^2$ given in terms of the bare coupling constant g_0 of the original 4d $U(N)$ field theory. We thus have the superpotential, given in terms of the A_i and B_i periods (1.6) as

$$-\frac{1}{2\pi i}W_{eff} = \sum_{i=1}^n N_i \Pi_i + \alpha \left(\sum_{i=1}^n S_i \right). \quad (4.6)$$

This W_{eff} depends on the coefficients g_r of the classical superpotential (1.1) of the original $U(N)$ theory with adjoint by way of the geometry (4.2). W_{eff} is a function of the n S_i , or equivalently the n unknown parameters in $f_{n-1}(x)$. The supersymmetric vacua have fixed $\langle S_i \rangle$, obtained by solving

$$\frac{\partial W_{eff}}{\partial S_i} = 0, \quad i = 1 \dots n. \quad (4.7)$$

These $\langle S_i \rangle$ will depend on the N_i , the parameters g_r entering in the original $W_{tree}(\Phi)$ and thus on the geometry (4.2), and Λ_0 , the B_i integral infrared cutoff.

In the classical limit, where we set the S_i to zero, and thus $f_{n-1}(x) = 0$, the period of the one-form (4.3) gives

$$\Pi_i = \frac{1}{2\pi i} \int_{a_i}^{\Lambda_0} dx W'(x) = \frac{1}{2\pi i} (W(\Lambda_0) - W(a_i)). \quad (4.8)$$

Then the dual superpotential is $W_{eff} = \sum_i N_i W(a_i)$ (ignoring the irrelevant constant $W(\Lambda_0)$). This indeed matches with the classical superpotential of the original $U(N)$ theory, given by simply evaluating the superpotential (1.1) in the vacuum with breaking (1.3), where N_i eigenvalues of the Φ field take eigenvalue a_i .

4.2. Aspects of the $U(1)^n$ gauge fields

The dual theory obtained after the transition is an $\mathcal{N} = 2$ $U(1)^n$ gauge theory, broken to $\mathcal{N} = 1$ $U(1)^n$ by the superpotential W_{eff} (4.6). The S_i , which are in the same $\mathcal{N} = 2$ multiplet as the $U(1)^n$, get masses and frozen to particular $\langle S_i \rangle$ by W_{eff} . On the other hand, the $\mathcal{N} = 1$ $U(1)^n$ gauge fields remain massless. The couplings τ_{ij} of these $U(1)$'s can be determined from $\Pi_i(S)$ or the $\mathcal{N} = 2$ prepotential $\mathcal{F}(S_i)$, with $\Pi_i = \partial \mathcal{F} / \partial S_i$, of the geometry under consideration:

$$\tau_{ij} = \frac{\partial \Pi_i}{\partial S_j} = \frac{\partial^2 \mathcal{F}(S_i)}{\partial S_i \partial S_j}. \quad (4.9)$$

The couplings (4.9) should be evaluated at the $\langle S_i \rangle$ obtained from (4.7).

Note that (4.7) and (4.9) imply

$$\sum_i N_i \tau_{ij} + \alpha = 0. \quad (4.10)$$

We identify the F_i the i -th block $U(1)$ field strength with the generator in $U(N)$ which is $1/N_i$ times the identity matrix in the i -th block and zero elsewhere. In this way the $F_i - F_j$ correspond to field strengths of the $U(1)^{n-1}$'s coming from the $SU(N)$ and the $N_i F_i$ will corresponds to the overall $U(1)$. Thus the above equation is consistent with the fact that the overall $U(1)$ is a linear combination of the $U(1)^n$'s with coefficients given by N_i , together with the fact that the bare coupling constant of the overall $U(1)$ should be the same as that of the original $U(N)$ theory, as the $U(1)$ is decoupled. Moreover it is consistent with the fact that there is no coupling between the field strength of this overall $U(1)$ with the other $U(1)^{n-1}$. Thus extremizing the superpotential is equivalent to this structure for the gauge coupling constants of the $U(1)$ factors.

One can also relate the coupling constants of the $U(1)$ factors to the period matrix of the hyperelliptic curve

$$y^2 = W'(x)^2 + f_{n-1}(x)$$

To see that, from (4.9) we will have to compute the period integrals of $\partial\omega/\partial(S_i - S_j)$ about the cycles of the hyperelliptic curve, where $\omega = ydx$. As we will discuss in section 7 the coefficient of x^{n-1} of $f_{n-1}(x)$ is proportional to the sum of S_i 's and thus considering $\partial\omega/\partial(S_i - S_j)$ gives rise to a linear combination of

$$\frac{x^{n-r} dx}{y}$$

with $2 \leq r \leq n$, a basis of the $n - 1$ holomorphic one-forms on the hyperelliptic curve. Thus τ_{ij} can be identified with the period matrix of the hyperelliptic curve.

4.3. Gauge theoretic reformulation

Just as in the case of $n = 1$ we can reformulate this duality in terms of a duality of two gauge systems: We start with $\mathcal{N} = 2$ pure Yang-Mills theory for gauge group $U(N)$ and deform it by the superpotential $W_{tree}(\Phi)$ of degree $n + 1$ in the scalar field, breaking the $U(N)$ into n factors $U(N_i)$. The $SU(N_i)$ gaugino bilinear together with the $U(1) \subset U(N_i)$ forms an $\mathcal{N} = 2$ multiplet. One considers a dual $\mathcal{N} = 2$ multiplet containing $U(1)^n$ softly

broken to $\mathcal{N} = 1$ by a superpotential term. Note that the $\mathcal{N} = 2$ we have proposed is of the form that appears in an $\mathcal{N} = 2$ theory with a $U(n)$ gauge group with some matter fields (whose structure is dictated by the superpotential). In fact the dual $\mathcal{N} = 2$ system we have been considering is of the type studied in [22] and was connected to a type IIB description considered here in [14]. In such a formulation the decoupling of the overall $U(1)$ from the other $U(1)$'s occurs as in (4.10), consistent with the minimization of the superpotential.

5. Field theory analysis

We now analyze the strong coupling dynamics of the $U(N)$ theory, with adjoint Φ and superpotential (1.1), in the vacuum with the classical breaking (1.3). In the quantum theory, each $\mathcal{N} = 1$ super Yang-Mills $SU(N_i)$ in (1.3) generally confines, with N_i supersymmetric vacua. The N_i vacua correspond to N_i -th roots of unity phases of the gaugino condensate $\langle S_i \rangle \neq 0$, with $S_i = -\frac{1}{32\pi^2} \text{Tr} W_\alpha W^\alpha$ the $SU(N_i)$ glueball chiral superfield. The $U(1)^n$ in (1.3) are free, and therefore remain unconfined and present in the low energy theory. The vacua can also have more interesting behavior. For example, in $SU(3)$ with a cubic superpotential for Φ but no quadratic mass term, the vacuum is at the non-trivial conformal field theory point of [22].

The low energy theory contains an effective superpotential $W_{eff}(g_p, \Lambda)$ which gives the chiral superfield expectation values via [23]

$$\begin{aligned} \frac{\partial W_{eff}(g_p, \Lambda)}{\partial g_p} &= \langle u_p \rangle \\ \frac{\partial W_{eff}(g_p, \Lambda)}{\partial \log \Lambda^{2N}} &= \langle S \rangle \equiv \sum_{i=1}^n \langle S_i \rangle. \end{aligned} \tag{5.1}$$

W_{eff} can often be obtained exactly, thanks to its holomorphic dependence on g_p and Λ [24]. In the present case, we'll discuss how W_{eff} can indeed, *in principle*, be obtained exactly via the $\mathcal{N} = 2$ curves [25,26,27]; in practice, however, the result is quite difficult to obtain.

5.1. Approximate W_{eff} via naive integrating in

The effective superpotential can often be obtained exactly via starting from the low-energy effective theory and “integrating in” the massive matter fields [23,28]. As discussed in [28], for this procedure to give an exact answer, one must be able to argue that the scale matching relations are known exactly and that a possible additional unknown contribution W_Δ to the superpotential necessarily vanishes. Our $\mathcal{N} = 1$ theory with adjoint Φ and superpotential (1.1), does not admit this kind of symmetry and limits arguments needed to prove the naive scale matching relations and $W_\Delta = 0$ as exact statements. So naive integrating in need not give the exact answer for W_{eff} ; nevertheless, it is still useful here for obtaining an approximate answer.

To illustrate how naive integrating in can fail to give the exact answer in the theory with adjoint Φ , consider the vacuum where classically $\langle \Phi \rangle = 0$, leaving $SU(N)$ unbroken. Such a vacuum exists for any tree level superpotential (1.1). The mass of Φ in this vacuum is $W''(0) = g_2 \equiv m$, independent of the other g_p . The low energy theory is $\mathcal{N} = 1$ $SU(N)$ pure Yang-Mills and the dynamical scale Λ_L of this theory is related to that of the original high energy theory by matching the running gauge coupling at the threshold scale m , giving $\Lambda_L^{3N} = m^N \Lambda^{2N}$. The low-energy theory has N vacua with gaugino condensation and low-energy superpotential

$$W_{low} = e^{2\pi i k/N} N \Lambda_L^3 = e^{2\pi i k/N} N m \Lambda^2. \quad (5.2)$$

Using (5.1) one could use this to try to find the $\langle u_r \rangle$ in this vacuum, but the answer would be incorrect for $SU(N)$ with $N > 3$. The exact answer can be found from deforming the $\mathcal{N} = 2$ curve following [29], as reviewed in the next subsection. The exact effective superpotential is found from this to be

$$W_{exact} = N \sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} \frac{g_{2p}}{2p} \Lambda^{2p} \binom{2p}{p}. \quad (5.3)$$

The g_2 term coincides with (5.2), so both give the same $\langle u_2 \rangle$, but (5.2) gives all other $\langle u_r \rangle = 0$, whereas (5.3) gives higher $\langle u_{2p} \rangle \sim N \Lambda^{2p} \neq 0$.

The terms in (5.3) which are missing from (5.2) are weighted by $g_{2p} \Lambda^{2p}$, which should be small as compared with the leading term $m \Lambda^2$. The reason is that the higher g_{2p} appear irrelevant in the original $SU(N)$ description, so their required UV cutoff should be larger than the dynamical scale Λ in order for the theory to be well-defined, i.e. the $g_{3+n} \Lambda^n$

should be small. So the lesson is that naive integrating in here needn't give the exact answer, but it does generally give the leading term or terms.

On the other hand, naive integrating in actually does give the *exact* answer for W_{eff} in the vacua where $SU(N) \rightarrow SU(2) \times U(1)^{N-2}$ [30]. In fact, the exact curve of the entire $\mathcal{N} = 2$ theory can be re-derived via “integrating in” in the $SU(2) \times U(1)^{N-2}$ vacua [30].

We now outline the naive integrating-in procedure for the general vacuum (1.3). The low-energy $\mathcal{N} = 1$ SYM with gauge group (1.3) leads to a low-energy superpotential via gaugino condensation in each of the decoupled, non-abelian groups:

$$W_{low} = W_{cl}(g_r) + \sum_{i=1}^n e^{2\pi i k_i / N_i} N_i \Lambda_i^3. \quad (5.4)$$

The term $W_{cl}(g_r)$ is simply the value of the classical superpotential (1.1), evaluated in the classical vacuum:

$$W_{cl} = \sum_{i=1}^n N_i \sum_{p=1}^{n+1} g_p \frac{a_i^p}{p}, \quad (5.5)$$

with the a_i defined in (1.2). As in (5.1), $\frac{\partial W_{cl}(g_r)}{\partial g_r} = \langle u_r \rangle_{cl}$.

The dynamical scale Λ_i entering in (5.4) is that of the low-energy $SU(N_i)$ theory, which is related to the scale Λ of the high-energy theory by matching the running gauge coupling across two thresholds: that of the massive $SU(N)/SU(N_i)$ W-bosons, and that of the mass of the field Φ in the vacuum. The classical masses of the W-bosons which are charged under $SU(N_i)$ are $m_{W_{ij}} = a_j - a_i$. The mass of the $SU(N_i)$ adjoint $\Phi_i \in \Phi$ is classically $m_{\Phi_i} = W''(a_i) = g_{n+1} \prod_{j \neq i} (a_j - a_i)$. The scale Λ_i of the low-energy $SU(N_i)$ is thus obtained by naive threshold matching to be

$$\Lambda_i^{3N_i} = \Lambda^{2N} m_{\Phi_i}^{N_i} \prod_{j \neq i} m_{W_{ij}}^{-2N_j} = g_{n+1}^{N_i} \Lambda^{2N} \prod_{j \neq i} (a_j - a_i)^{N_i - 2N_j}. \quad (5.6)$$

It will be useful in what follows to also integrate in the glueball fields S_i :

$$W_{low} = W_{cl}(g_r) + \sum_{i=1}^n S_i \left(\log\left(\frac{\Lambda_i^{3N_i}}{S_i^{N_i}}\right) + N_i \right). \quad (5.7)$$

The S_i are massive, with supersymmetric vacua $\langle S_i \rangle = \Lambda_i^{3N_i}$, and integrating out the S_i leads back to (5.4).

The final result of naive integrating in is thus expressed in terms of the $a_i(g_r)$ as

$$W_{low}(g_r) = \sum_{i=1}^n \left[N_i \sum_{p=1}^{n+1} g_p \frac{a_i^p}{p} + S_i \left(\log \left(\frac{g_{n+1}^{N_i} \Lambda^{2N} \prod_{j \neq i} (a_j - a_i)^{(N_i - 2N_j)}}{S_i^{N_i}} \right) + N_i \right) \right]. \quad (5.8)$$

The quantum term in (5.8), coming from $SU(N_i)$ gaugino condensation, is to be omitted when $N_i = 1$; e.g. in the case of [30], where $N_1 = 2$ and all other $N_i = 1$. The result (5.8) happens to be exact when no $N_i > 2$ but, as emphasized above, (5.8) is only an approximation to the exact answer in the more general case, where some $N_i \geq 3$.

5.2. The exact $W_{exact}(g_r)$ via deforming the $\mathcal{N} = 2$ results

In this subsection, we obtain the exact 1PI generating function $W_{exact}(g_r)$ by deforming the exact solution [25,26,27] of the $\mathcal{N} = 2$ theory by the $W_{tree}(\Phi)$ (1.1). The large N duality proposal of section 4 gives the exact superpotential $W_{exact}(g_r; S_i)$ as (4.6), with the glueball fields included. (As verified in section 7, the naive integrating in result (5.8) is indeed an approximation to this exact result; generally there is an infinite series expansion of corrections to the naive formula (5.8).) Upon integrating out the massive S_i from $W_{exact}(g_r, S_i)$ (4.6), one obtains $W_{exact}(g_r)$, which we will verify indeed agrees with the field theory result obtained in this subsection. Our $W_{exact}(g_r; S_i)$ (4.6), however, contains the additional information about the glueball fields S_i . Although the S_i are massive, this additional information about their superpotential is physical; for example ΔW between the different $\langle S_i \rangle$ vacua gives the BPS tension of the associated domain walls. Perhaps there's also a way to exactly integrate in the S_i in the context of the deformed $\mathcal{N} = 2$ field theory, though this is not presently known.

The $\mathcal{N} = 2$ theory deformed by $W_{tree} = \sum_{i=1}^{n+1} g_r u_r$ only has unbroken supersymmetry on submanifolds of the Coulomb branch, where there are additional massless fields besides the u_r . The additional massless fields are the magnetic monopoles or dyons, which become massless on some particular submanifolds $\langle u_p \rangle$ [25]. Near a point with l massless monopoles, the superpotential is

$$W = \sum_{k=1}^l M_k(u_r) q_k \tilde{q}_k + \sum_{p=1}^{n+1} g_p u_p, \quad (5.9)$$

and the supersymmetric vacua are at those $\langle u_p \rangle$ satisfying

$$M_k(\langle u_p \rangle) = 0 \quad \text{and} \quad \sum_{k=1}^l \frac{\partial M_k(\langle u_p \rangle)}{\partial u_p} \langle q_k \tilde{q}_k \rangle + g_p = 0, \quad (5.10)$$

the first equations are for all $k = 1 \dots l$ and the second for all $r = 1 \dots N$ (with $g_p = 0$ for $p > n + 1$). The value of the superpotential (5.9) in this vacuum is simply

$$W_{eff} = \sum_{p=1}^{n+1} g_p \langle u_p \rangle, \quad (5.11)$$

with $\langle u_p \rangle$ the solution of $M_k(\langle u_p \rangle) = 0$, where the monopoles are massless. The explicit monopole masses $M_k(u_r)$ on the Coulomb branch can be obtained via the appropriate periods of the one-form [26],

$$M_k = \oint_{\gamma_k} \lambda, \quad \text{with} \quad \lambda = \frac{1}{2\pi i} \frac{x}{y} \frac{\partial P_N(x)}{\partial x} dx, \quad \text{satisfying} \quad \frac{\partial \lambda}{\partial s_r} \sim \frac{x^{N-r} dx}{y} + d(\dots), \quad (5.12)$$

but this will not be needed here.

In the vacuum (1.3), there are n massless photons, whereas the original $\mathcal{N} = 2$ theory had N massless photons. So the vacuum (1.3) must have $N - n$ mutually local magnetic monopoles being massless and getting an expectation value as in (5.10), $\langle q_k \tilde{q}_k \rangle \neq 0$ for $k = 1 \dots N - n$. It can indeed be shown from (5.10) that if the highest Casimir with nonzero g_p in W_{tree} is u_{n+1} , as in (1.1), then the supersymmetric vacuum necessarily has at least $l = N - n$ mutually local monopoles condensed. (More than $N - n$ condensed monopoles correspond to those classical vacua in (1.3) where some $N_i = 0$, and thus there are fewer than n photons left massless.) The vacuum obtained from integrating out u_p as in (5.10), will give some values of the $\langle u_p \rangle$ which are determined in terms of the g_p .

Solving for the supersymmetric vacua as in (5.10), is equivalent to minimizing $W_{tree} = \sum_{p=1}^{n+1} g_p u_p$, subject to the constraint that $\langle u_p \rangle$ lie on the the codimension $N - n$ subspace of the Coulomb branch where at least $N - n$ mutually local monopoles or dyons are massless. This is just a matter of replacing the monopoles with $N - n$ Lagrange multipliers, imposing that the u_r lie in the subspace with $N - n$ massless monopoles; i.e. we integrate out the u_p with $W = W_{tree} + \sum_{k=1}^{N-n} L_k M_k(u)$, with $M_k(u)$ the monopole masses on the Coulomb branch and L_k Lagrange multipliers, and the $\langle L_k \rangle = \langle q_k \tilde{q}_k \rangle$. The resulting $\langle u_p \rangle$ will be some fixed value, depending on the g_r and Λ , giving finally $W_{exact}(g_r, \Lambda) = \sum_r g_r \langle u_r \rangle$.

Recall that the curve of the $U(N)$ theory is

$$y^2 = P(x; u_r)^2 - 4\Lambda^{2N}, \quad P(x, u_r) \equiv \det(x - \Phi) = \sum_{k=0}^N x^{N-k} s_k, \quad (5.13)$$

with the s_k related to the u_r by

$$ks_k + \sum_{r=1}^k ru_r s_{k-r} = 0, \quad (5.14)$$

and $s_0 \equiv 1$ and $u_0 \equiv 0$; thus $s_1 = -u_1$, $s_2 = \frac{1}{2}u_1^2 - u_2$, etc. (for $SU(N)$ we impose $u_1 = 0$). The condition for having $N - n$ mutually local massless magnetic monopoles is that

$$P_N(x; \langle u_p \rangle)^2 - 4\Lambda^{2N} = (H_{N-n}(x))^2 F_{2n}(x), \quad (5.15)$$

where H_{N-n} is a polynomial in x of degree $N - n$ and F_{2n} is a polynomial in x of degree $2n$. The LHS of (5.15) has $2N$ roots, and the RHS says that $N - n$ pairs of roots should be tuned to coincide; thus (5.15) is satisfied on codimension $N - n$ subspaces of the Coulomb branch. We need to integrate out the u_p , with $W_{tree} = \sum_{p=1}^{n+1} g_p u_p$, subject to the constraint that $\langle u_p \rangle$ satisfy (5.15).

Of the n massless photons, the one corresponding to the trace of $U(N)$, does not couple to the rest of the theory and so its coupling constant is the same as the one we started with. The other $n - 1$ photons which are left massless in (1.3) have gauge couplings which are given by the period matrix of the reduced curve

$$y^2 = F_{2n}(x; \langle u_r \rangle) = F_{2n}(x; g_p, \Lambda), \quad (5.16)$$

with $F_{2n}(x; \langle u_p \rangle)$ the same function appearing in (5.15) and $\langle u_r \rangle$ the point on the solution space of (5.15) which minimizes W_{tree} . The curve (5.16) thus gives the exact gauge couplings of $U(1)^{n-1}$ which remain massless in (1.3) as functions of g_p and Λ .

The dual Calabi-Yau geometry which we proposed in section 4,

$$W'(x)^2 + f_{n-1}(x) + y^2 + z^2 + v^2 = 0,$$

is already similar to the SW geometry (5.16), giving the coupling constants of the massless $U(1)$'s. To show that the τ_{ij} obtained from (5.16) agrees with that obtained from (4.9), we need to show that the $F_{2n}(x)$ of (5.15) and (5.16) is given by

$$g_{n+1}^2 F_{2n}(x) = W'(x)^2 + f_{n-1}(x), \quad (5.17)$$

with the factor of g_{n+1}^2 because the highest order term in $F_{2n}(x)$ is x^{2n} , whereas that of $W'(x)$ is $g_{n+1}x^n$. We will indeed verify that the structure of F_{2n} predicted from (5.17) is

correct, i.e. it is a deformation of a degree $n - 1$ polynomial in x added to W'^2 . However more needs to be done to show that the dual geometry and gauge theory predict the same coupling constants for the $U(1)$'s. Namely, we have to show that the coefficients of the f_{n-1} predicted from dual geometry and that of the gauge theory have identical dependence on N_i and the parameters of the superpotential. This is indeed a highly non-trivial statement, which we will later verify for cubic superpotential in section 8.

As a first hint about why (5.17) holds, consider the classical limit, $\Lambda \rightarrow 0$, where $P_N(x) = \det(x - \Phi) \rightarrow \prod_{i=1}^n (x - a_i)^{N_i}$, with a_i the roots of $W'(x) = g_{n+1} \prod_{i=1}^n (x - a_i)$. In this limit $P_N^2 - 4\Lambda^{2N} \rightarrow H_{N-n}^2 F_{2n}$, as in (5.15), with $H_{N-n}(x) = \prod_{i=1}^n (x - a_i)^{N_i-1}$ and $F_{2n} = \prod_{i=1}^n (x - a_i)^2 = g_{n+1}^{-2} W'(x)^2$. The motivation for this splitting is applying the intuition of [29] to each $SU(N_i)$ factor: each $P_{N_i}^2 - 1$ splits to $(x - a_i)^2$ times a degree $N_i - 1$ polynomial. We thus find that (5.17) holds in the $\Lambda \rightarrow 0$ limit, and see that the $f_{n-1}(x)$ appearing in (5.17) satisfies $f_{n-1}(x) \rightarrow 0$ for $\Lambda \rightarrow 0$.

To prove (5.17) exactly, and also get some insight into how the $\langle u_r \rangle$ are determined, we note that we can minimize our W_{tree} (1.1), subject to the constraint that the $\langle u_r \rangle$ satisfy (5.15), by introducing several Lagrange multipliers:

$$W = \sum_{r=1}^n g_r u_r + \sum_{i=1}^l [L_i(P_N(x; u_r))|_{x=p_i} - 2\epsilon_i \Lambda^N] + Q_i \frac{\partial}{\partial x} P_N(x; u_r)|_{x=p_i}, \quad (5.18)$$

with $\epsilon_i = \pm 1$. We're generally allowing l mutually local massless monopoles, and will see that $l \geq N - n$. The L_i , Q_i , and p_i are all treated as Lagrange multipliers; so we should independently take derivatives of (5.18) with respect to all u_r , L_i , Q_i , and p_i , and set all these derivatives to zero. The p_i will be the roots of $H_l(x)$ in (5.15), and the L_i and Q_i constraints implement the LHS of (5.15) having double zeros at these l points p_i .

The variation of (5.18) with respect to p_i gives

$$Q_i \frac{\partial^2 P_N}{\partial x^2} \Big|_{x=p_i} = 0, \quad (5.19)$$

where we used the Q_i constraint to eliminate the term involving L_i . For generic g_r , the RHS of (5.15) has some double roots, but no triple or higher roots; therefore (5.19) implies that $\langle Q_i \rangle = 0$. The situation where the RHS of (5.15) does have triple or higher order roots is where the unperturbed $\mathcal{N} = 2$ theory has an interacting $\mathcal{N} = 2$ superconformal field theory, as in [22]. Our $\mathcal{N} = 1$ theory with W_{tree} does put the vacuum at such points

for some special choices of the g_r , but we'll consider the generic situation for the moment. Since the $\langle Q_i \rangle = 0$, the variation of (5.18) with respect to all u_r gives

$$g_r + \sum_{i=1}^l \sum_{j=0}^N L_i p_i^{N-j} \frac{\partial s_j}{\partial u_r} = 0, \quad (5.20)$$

with the understanding that the $g_r = 0$ for $r > n + 1$. Using (5.14), (5.20) becomes

$$g_r = \sum_{i=1}^l \sum_{j=0}^N L_i p_i^{N-j} s_{j-r}. \quad (5.21)$$

We should also impose the L_i and Q_i constraints in (5.18). These equations and (5.21) fix the $\langle u_r \rangle$, $\langle L_i \rangle$, $\langle p_i \rangle$, and $\langle Q_i \rangle$ as functions of the g_r and Λ . The $\langle L_i \rangle$ are proportional to the expectation values $\langle q_i \tilde{q}_i \rangle$ of the $l \geq N - n$ condensed, mutually local, monopoles.

Following a similar argument in [31], we multiply (5.21) by x^{r-1} and sum:

$$\begin{aligned} W'_{cl}(x) &= \sum_{r=1}^N g_r x^{r-1} \\ &= \sum_{r=1}^N \sum_{i=1}^l \sum_{j=0}^N x^{r-1} p_i^{N-j} s_{j-r} L_i \\ &= \sum_{r=-\infty}^N \sum_{i=1}^l \sum_{j=0}^N x^{r-1} p_i^{N-j} s_{j-r} L_i - 2L\Lambda^N x^{-1} + \mathcal{O}(x^{-2}) \\ &= \sum_{i=1}^l \sum_{j=-\infty}^N P_N(x; \langle u \rangle) x^{j-N-1} p_i^{N-j} L_i - 2L\Lambda^N x^{-1} + \mathcal{O}(x^{-2}) \\ &= \sum_{i=1}^l \frac{P_N(x; \langle u \rangle)}{x - p_i} L_i - 2L\Lambda^N x^{-1} + \mathcal{O}(x^{-2}). \end{aligned} \quad (5.22)$$

We define $L \equiv \sum_{i=1}^l L_i \epsilon_i$. Defining, as in [31], the order $l - 1$ polynomial $B_{l-1}(x)$ by

$$\sum_{i=1}^l \frac{L_i}{x - p_i} = \frac{B_{l-1}(x)}{H_l(x)}, \quad (5.23)$$

with $H_l(x)$ the polynomial appearing in (5.15), we thus have

$$W'_{cl}(x) + 2L\Lambda^N x^{-1} = B_{l-1}(x) \sqrt{F_{2N-2l}(x) + \frac{4\Lambda^{2N}}{H_l(x)^2}} + \mathcal{O}(x^{-2}). \quad (5.24)$$

Since the highest order term in W'_{cl} is $g_{n+1}x^n$, we see that $B_{l-1}(x)$ should actually be order $n - N + l$. This shows that $l \geq N - n$ and, in particular, for $l = N - n$, $B_{N-n-1} = g_{n+1}$ is a constant. Squaring (5.24) gives

$$g_{n+1}^2 F_{2n} = W_{cl}'^2 + 4g_{n+1}L\Lambda^N x^{n-1} + \mathcal{O}(x^{n-2}). \quad (5.25)$$

We have thus derived (5.17), $g_{n+1}^2 F_{2n} = W'^2 + f_{n-1}(x)$, and found that $f_{n-1}(x) = 4g_{n+1}L\Lambda^N x^{n-1} + \mathcal{O}(x^{n-2})$.

This shows that the exact $\tau_{ij}(g_r, \Lambda)$ of the $U(1)^n$ photons left massless found using the reduced $\mathcal{N} = 2$ curve (5.16), evaluated in the supersymmetric vacua, is consistent with that of (4.9), found in section 4 via our large N duality. However as noted before to show they are exactly the same we have to match the coefficients of $f_{n-1}(x)$, which depends in a highly non-trivial way on N_i and the coupling constants of the superpotential. The above method also, in principle, gives the $\langle u_r \rangle$, and thus $W_{eff}(g_r)$, which can be compared with the duality result $W_{exact}(g_r, S_i)$ (4.6) (upon integrating out the S_i). The duality results (4.9) and (4.6) give the answers, and in particular the N_i dependence, in a much simpler and more elegant fashion.

It is interesting to ask if the duality results of section 4 could be recovered more directly by a field theory analysis which includes the n glueball chiral superfields S_i of the unbroken gauge group $\prod_{i=1}^n U(N_i)$. In the original $SU(N)$ theory, we can construct N generalized glueball objects $\sim \text{Tr} \Phi^i W_\alpha W^\alpha$, $i = 0 \dots N - 1$. The $N - n$ monopole condensates or Lagrange multiplier expectation values in the above analysis is (indirectly) related to $N - n$ of these generalized glueballs. The n remaining ones should be those of the unbroken low-energy $\prod_{i=1}^n U(N_i)$. It is not known how to exactly include these from a direct field theory analysis.

For any W_{tree} , there are vacua where classically $U(N)$ or $SU(N)$ is unbroken and, in the quantum theory, $N - 1$ mutually local monopoles condense. These are the only vacua for $W_{tree} = mu_2$, but also exist for any $n \geq 1$. The condition for having the $N - 1$ mutually local massless monopoles is [29]

$$P(x; \langle u_r \rangle)^2 - 4\Lambda^{2N} = H_{N-1}(x)^2 F_2(x), \quad (5.26)$$

which is satisfied via Chebyshev polynomials:

$$P_N(x, \langle u_r \rangle) = \Lambda^N T_N\left(\frac{x}{\Lambda}\right); \quad T_N(x \equiv t + t^{-1}) = t^N + t^{-N}. \quad (5.27)$$

With the normalization of (5.27), $T_N(x) = x^N - Nx^{N-2} + \dots$, the first Chebyshev polynomials. The roots of $P_N = \det(x - \Phi)$, as given by (5.27), are $\phi_j = 2\Lambda \cos((2j+1)\pi/2N)$, $j = 0 \dots N-1$; this gives (5.3).

More generally, we can use Chebyshev polynomials to construct new solutions of the massless monopoles constraint (5.15). Given a solution $P_N(x)$ of (5.15) which is appropriate for the $SU(N)$ theory where the vacuum is broken to

$$SU(N) \rightarrow \otimes_{i=1}^n SU(N_i) \otimes U(1)^{N-1} \quad \text{with} \quad \sum_i N_i = N, \quad (5.28)$$

we can immediately construct the solution $P_{KN}(x)$ which is appropriate for a $SU(KN)$ theory, with the same W_{tree} (1.1), in the vacuum where the gauge group is broken as

$$SU(KN) \rightarrow \otimes_{i=1}^n SU(KN_i) \otimes U(1)^{N-n} \quad \text{with} \quad \sum_i N_i = N. \quad (5.29)$$

The solution $P_{KN}(x)$ of (5.15) for the theory (5.29) is given by the Chebyshev polynomial of the $K=1$ solution $P_N(x)$:

$$P_{KN}(x) = \tilde{\Lambda}^{NK} T_K \left(\frac{P_N(x)}{\Lambda^N} \right), \quad (5.30)$$

with $\tilde{\Lambda}$ and Λ the scales of $SU(KN)$ and $SU(N)$, respectively. To see that this satisfies the condition of (5.15) note

$$\begin{aligned} P_{KN}(x)^2 - 4\tilde{\Lambda}^{2KN} &= \tilde{\Lambda}^{2NK} \left(T_K \left(\frac{P_N}{\Lambda^N} \right)^2 - 4 \right) = \tilde{\Lambda}^{2KN} [U_{K-1} \left(\frac{P_N}{\Lambda^N} \right)]^2 \left(\frac{P_N^2}{\Lambda^{2N}} - 4 \right) \\ &= \tilde{\Lambda}^{2KN} \Lambda^{-2N} [U_{K-1} \left(\frac{P_N}{\Lambda^N} \right) H_{N-n}(x)]^2 F_{2n}(x) \equiv [H_{KN-n}(x)]^2 F_{2n}(x). \end{aligned} \quad (5.31)$$

We denote the second Chebyshev functions $U_{K-1}(x \equiv t + t^{-1}) \equiv (t^K - t^{-K})/(t - t^{-1}) = x^{K-1} + \dots$, and the second line uses the fact that P_N is a solution of (5.15). Thus $P_{KN}(x)$ given by (5.30) indeed satisfies the condition (5.15) appropriate for (5.29). Furthermore, the $U(1)^{N-n}$ in (5.29) has gauge couplings given by the curve $y^2 = F_{2n}(x)$, which is the same as that of the $K=1$ theory. This fits with the dual geometry prediction of section 4, as will be discussed in the next section.

Expanding out (5.30) relates the expectation values $\langle \tilde{u}_p \rangle$ of the $SU(KN)$ theory to the $\langle u_p \rangle$ of the $SU(N)$ theory. The relation is especially simple for the lower Casimirs:

$$\tilde{u}_2 = K u_2, \quad \tilde{u}_3 = K u_3, \quad (5.32)$$

with some more complicated relations for the general higher Casimirs.

By the above construction, it suffices to consider (1.3) where the N_i have no common integer divisor. The simple K dependence fits with the duality results of section 4.

5.3. Other possible connections

The quantum $\mathcal{N} = 2$ theory is related to an integrable hierarchy, which is known to have integrable “Whitham hierarchy deformations;” see e.g. [32]. Our superpotential W_{tree} is naturally regarded as a Whitham deformation of the $\mathcal{N} = 2$ theory, where the Whitham “times” are the g_r in (1.1) which, from the $\mathcal{N} = 2$ perspective, are spurions breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$. The exact solution can still be obtained as a Θ function of the Whitham hierarchy, see e.g. the last reference of [32]. It would be interesting to see how this Θ function is related to the S_i and Π_i periods of section 4.

The $\mathcal{N} = 1$ $U(N)$ field theories with adjoint Φ , N_f fundamental flavors, and general superpotential $W_{tree}(\Phi)$ (1.1) can also be constructed via N IIA D4 branes suspended between a NS brane and n NS’ branes. The construction was discussed in detail in [31] and references cited therein. Four of the five directions transverse to the D4s in IIA are conventionally written as having complex coordinates w and v . The NS’ branes are given by some (v, w) curve, which classically is $w = W'_{tree}(v)$, giving the n NS’ branes at the minima of W_{tree} . Going to M-theory, the brane configuration becomes a smooth M5 brane configuration, as in [33]. Our geometric flop transition duality is roughly reminiscent of exchanging the roles of v and w ; it was already speculated [31] that this exchange could be related to the field theory duality of [9]. Perhaps this can be made more precise.

6. The case with the cubic superpotential in more detail

Consider in more detail the case $n = 2$, with $W_{cl} = gu_3 + mu_2 + \lambda u_1$. Then $W' = g(\phi - a_1)(\phi - a_2)$, with

$$a_1 = \frac{m}{2g} + \sqrt{\left(\frac{m}{2g}\right)^2 - \frac{\lambda}{g}}, \quad a_2 = \frac{m}{2g} - \sqrt{\left(\frac{m}{2g}\right)^2 - \frac{\lambda}{g}}. \quad (6.1)$$

For $SU(N) \rightarrow SU(N_1) \times SU(N_2) \times U(1)$, as opposed to $U(N) \rightarrow U(N_1) \times U(N_2)$, λ should be treated as a Lagrange multiplier, enforcing $u_1 = 0$. In that case,

$$a_1 = \left(\frac{m}{g}\right) \frac{N_2}{(N_1 - N_2)}, \quad a_2 = -\left(\frac{m}{g}\right) \frac{N_1}{(N_1 - N_2)}. \quad (6.2)$$

The classical low-energy superpotential is

$$W_{cl} = \frac{m^3}{g^2} \cdot \frac{N_1 N_2 (N_1 + N_2)}{6(N_1 - N_2)^2} \quad (6.3)$$

and

$$\Lambda_1^{3N_1} = g^{N_1} \Delta^{N_1-2N_2} \Lambda^{2N} \quad \Lambda_2^{3N_2} = g^{N_2} \Delta^{N_2-2N_1} \Lambda^{2N}, \quad (6.4)$$

with $m_W = a_1 - a_2 = (m/g)(N/(N_1 - N_2)) \equiv \Delta$ and $m_\phi = g\Delta$. Naive “integrating in” then gives $W_{eff} = W_{cl} + W_{np}$ with

$$\begin{aligned} W_{np} &= \sum_{i=1}^2 S_i \left(\log\left(\frac{\Lambda_i^{3N_i}}{S_i^{N_i}}\right) + N_i \right) \\ &= N_1 \left[S_1 \log\left(\frac{g\Lambda^2\Delta}{S_1}\right) + S_1 + S_2 \log\left(\frac{\Lambda^2}{\Delta^2}\right) \right] + N_2 \left[S_2 \log\left(\frac{g\Lambda^2\Delta}{S_2}\right) + S_2 + S_1 \log\left(\frac{\Lambda^2}{\Delta^2}\right) \right]. \end{aligned} \quad (6.5)$$

The exact answer for the *value* of the superpotential at the minima of W can be obtained via deforming the $\mathcal{N} = 2$ curve, is given by (5.11), with the $\langle u_r \rangle$ given by solving (5.15) for $n = 2$:

$$P_N^2 - 4\Lambda^{2N} = H_{N-2}^2 F_4. \quad (6.6)$$

Again, this does not include the glueball fields.

As discussed in the previous section, a solution of (6.6) appropriate for $SU(N) \rightarrow SU(N_1) \times SU(N_2) \times U(1)$ can be used to immediately construct a solution of (6.6) appropriate for $SU(KN) \rightarrow SU(KN_1) \times SU(KN_2) \times U(1)$. Using (5.32), the low energy effective superpotential for the $SU(KN)$ theory is

$$W_{eff}[SU(KN)] = m\langle \tilde{u}_2 \rangle + g\langle \tilde{u}_3 \rangle = Km\langle u_2 \rangle + Kg\langle u_3 \rangle = KW_{eff}[SU(N)], \quad (6.7)$$

simply a factor of K times that of the $SU(N)$ theory. The $\langle c_l \rangle$ which minimizes W_{eff} , giving the vacuum on the solution space of (6.6), is thus K independent, so K really does just factor out as an overall multiplicative factor in the superpotential.

6.1. Examples:

$$U(3N) \rightarrow U(2N) \times U(N)$$

As a simple example of the procedure outlined in the last section, consider the case of $U(3N)$ in the vacuum where the unbroken group is $U(2N) \times U(N)$. As discussed above it suffices to consider the case $N = 1$. The superpotential of (5.18) is

$$W = \lambda u_1 + mu_2 + gu_3 + L(p^3 + s_1 p^2 + s_2 p + s_3 \pm 2\Lambda^3) + Q(3p^2 + 2s_1 p + s_2). \quad (6.8)$$

The p equation of motion (along with Q 's) gives $\langle Q \rangle = 0$ and (5.21) then gives $\lambda = L(p^2 + ps_1 + s_2)$, $m = L(p + s_1)$, $g = L$. Thus $\langle s_1 \rangle = g^{-1}m - p$, $\langle s_2 \rangle = g^{-1}(\lambda - mp)$, and $\langle s_3 \rangle = \mp 2\Lambda^3 - pg^{-1}\lambda$. $\langle p \rangle$ is fixed by the Q constraint to be either a_1 or a_2 of (6.1), so $W'_{cl}(x) = g(x-p)(x+p+g^{-1}m)$. We then have $\langle P_3(x) \rangle = g^{-1}(x-p)W'_{cl}(x) \mp 2\Lambda^3$, and thus $P_3^2 - 4\Lambda^6 = (x-p)^2 F_4(x)$, with $g^2 F_4(x) = W'(x)^2 \mp 4g\Lambda^3(gx+gp+m)$, which matches with (5.25). For $SU(3)$, we treat λ also as a Lagrange multiplier, enforcing $\langle s_1 \rangle = -\langle u_1 \rangle = 0$, i.e. $\langle p \rangle = m/g$. The Q constraint then gives $\langle \lambda \rangle = -2m^2/g$, so $\langle u_2 \rangle = 3(m/g)^2$ and $\langle u_3 \rangle = -2(m/g)^3 \pm 2\Lambda^3$. Plugging these back into W gives $W_{low} = (m^3/g^2) \pm 2g\Lambda^3$.

Equivalently, we could simply solve the L and Q constraints at the outset by taking $P_3 = (x-a)^2(x-b) \mp 2\Lambda^3$, giving $\langle u_1 \rangle = 2a+b$, $\langle u_2 \rangle = 2a^2+b^2$, $\langle u_3 \rangle = 2a^3+b^3 \pm 2\Lambda^3$ and thus $W_{low} = 2W(a) + W(b) \pm 2g\Lambda^3$. Minimizing with respect to a and b gives $\langle a \rangle = a_1$, $\langle b \rangle = a_2$ and $W_{low} = W_{cl} \pm 2g\Lambda^3$ with $W_{cl} = 2W(a_1) + W(a_2)$. In order to get the $SU(3) \rightarrow SU(2) \times U(1)$ answer we impose $\partial W_{low}/\partial \lambda = 0$, which implies $a_1 = \frac{m}{g}$, $a_2 = -2\frac{m}{g}$.

We thus find for $SU(3)$ $W_{low} = (m^3/g^2) \pm 2g\Lambda^3$ and the remaining massless photon has gauge coupling $\tau(g\Lambda/m)$ which is given exactly by the curve $y^2 = g^2 F_4(x) = W'^2 \mp 4g\Lambda^3(gx+2m)$, with $W'(x) = g(x-\frac{m}{g})(x+2\frac{m}{g})$. This curve degenerates at $(m/g)^3 = \pm \Lambda^3$, i.e. $\langle u_3 \rangle = 0$, which is where an additional magnetic monopole becomes massless in the $\mathcal{N} = 2$ theory. The $SU(2)$ glueball has $\langle S \rangle = \pm g\Lambda^3$.

Splittings of $SU(5)$

The computation of the one parameter family of $\mathcal{N} = 2$ curves for the different splittings of $SU(5)$, namely, $SU(3) \times SU(2) \times U(1)$ and $SU(4) \times U(1)$ can be done explicitly. This will provide the highly non-trivial exact answer for the low energy effective superpotential that will be used to check the answer from the geometry in section 8.4. As discussed before this answer also provides the solution for $SU(5K) \rightarrow SU(3K) \times SU(2K) \times U(1)$ and $SU(5K) \rightarrow SU(4K) \times SU(K) \times U(1)$ for any integer K .

We need to solve (6.6) for $N = 5$, i.e. to find $P_5(x)$ such that

$$P_5^2(x) - 4\Lambda^{10} = F_4(x)H_3^2(x) \quad (6.9)$$

Clearly, $P_5(x)$ has five parameters, given by the positions of the roots since the coefficient of x^5 can be normalized to one. However, three of them have to be used to produce the

three double roots and one in order to impose the quantum tracelessness condition, i.e., to set to zero the x^4 coefficient. This leaves us with a one parameter family of curves.

Let us set $\Lambda^5 = \frac{1}{2}$ and $H_3(x) = (x-a)(x-b)x$. The LHS of (6.9) can be factored as $(P_5 - 1)(P_5 + 1)$ where it is clear that the two factors should contain no common roots. Therefore we can freely set,

$$P_5(x) = (x-a)^2(x-b)^2(x-c) \mp 1 \quad (6.10)$$

Now we want to make sure that $P_5 \mp 1$ will have a double root at $x = 0$. This condition can be easily implemented by,

$$P_5(0) = \pm 1 \quad \frac{dP_5}{dx}(0) = 0$$

In terms of a, b and c , these conditions read as follows,

$$a^2b^2c = \pm 2 \quad ab(2c(a+b) + ab) = 0 \quad (6.11)$$

Finally, we can impose the tracelessness condition by shifting $x \rightarrow x - \frac{1}{5}(2(a+b) + c)$. We can now read off the gauge theory Casimir expectation values (using $\langle Tr\Phi \rangle = 0$),

$$P_5(x) = \langle det(x - \Phi) \rangle = x^5 - \frac{1}{2}\langle Tr\Phi^2 \rangle x^3 - \frac{1}{3}\langle Tr\Phi^3 \rangle x^2 + \dots$$

Since, our solution is symmetric in a and b it is more natural to write it in term of the symmetric polynomials $s = a + b$ and $k = ab$. The constraints (6.11) now read $k^2c = \pm 2$ and $k(2cs + k) = 0$. Assuming that $k \neq 0$ we can solve for k as $k = -2cs$. Then we are left with only one constraint, namely, $2s^2c^3 = \pm 1$.

The Casimirs are now given by,

$$u_2 = \frac{1}{5}(2c^2 + 18cs + 3s^2) \quad u_3 = -\frac{2}{25}(2c^3 - 23c^2s + 9cs^2 + s^3)$$

and the superpotential is now a function of c or s depending on how we use the constraint. Let us introduce the constraint through a Lagrange multiplier β and write the superpotential as,

$$W_{eff}(c, s, \beta) = gu_3(c, s) + mu_2(c, s) + \beta(\pm\Lambda^5 - s^2c^3)$$

where we have introduced Λ back for later convenience.

Now we need to solve $\frac{\partial W_{eff}}{\partial c} = 0$ and $\frac{\partial W_{eff}}{\partial s} = 0$ and then impose the constraint. Computing these two equations and using one of them to eliminate β from the other we get the following simple equation,

$$3c + s = \frac{5m}{g} \quad (6.12)$$

subject to the constraint $s^2 c^3 = \pm \Lambda^5$. There is yet a better way to write the constraint, namely, $s^4 c^6 = \Lambda^{10}$. This will make very simple the identification of the different vacua.

Now we can see how the different splittings will come out. The classical limit corresponds to setting $\Lambda \rightarrow 0$ and the constraint can be solved in two ways, namely, $s = 0$ or $c = 0$. The former leads to $c = \frac{5m}{3g}$ using (6.12) while the latter leads to $s = \frac{5m}{g}$. Plugging this in the superpotential we reproduce in the former case the classical answer for $SU(4) \times U(1)$ and in the latter we get that of $SU(3) \times SU(2) \times U(1)$.

$$SU(4) \times U(1)$$

In order to get W_{low} we need to solve for c using (6.12) and $s^4 = \frac{\Lambda^{10}}{c^6}$. Clearly, we have 4 solutions to the constraint giving $s = s(c)$. These are the $N_1 N_2 = 4$ vacua. The equation we need to solve is then

$$c = \frac{5m}{3g} - \frac{s(c)}{3}$$

this can be solved recursively using $t \equiv (\frac{3g\Lambda}{5m})^{5/2}$ as expansion parameter. Once this is done, s can also be found and plugging them back in the superpotential we get,

$$W_{low} = \frac{125}{27} \frac{m^3}{g^2} \left(\frac{2}{25} + 4t - \frac{1}{3}t^2 - \frac{7}{54}t^3 - \frac{5}{54}t^4 - \frac{221}{2592}t^5 - \frac{22}{243}t^6 + \right. \\ \left. - \frac{2185}{20736}t^7 - \frac{286}{2187}t^8 - \frac{9147325}{53747712}t^9 + \dots \right)$$

The above exact answer for the value of the superpotential at the critical point differs from the naive integrating in analysis (5.4), which would terminate at order t^2 . The coefficients of the classical t^0 term and t term agree with the exact answer above, but the coefficient of t^2 term differs from the exact answer.

$$SU(3) \times SU(2) \times U(1)$$

In this case we need to solve for s using $c^6 = \frac{\Lambda^{10}}{s^4}$. Here, we have 6 solutions giving the $N_1 N_2 = 6$ choices of vacua. The equation in this case becomes,

$$s = \frac{5m}{g} - 3c(s)$$

solving as before but using as expansion parameter $t \equiv (\frac{g\Lambda}{5m})^{5/3}$ we get for the superpotential the following expression,

$$W_{low} = \frac{250}{2} \frac{m^3}{g^2} \left(\frac{1}{25} + 3t^2 + 2t^3 + 6t^4 + 26t^5 + 135t^6 + 782t^7 + \frac{14630}{3}t^8 + 32076t^9 + \dots \right)$$

Again this differs from the result of the naive low energy analysis (5.4) which would terminate at order t^3 ; up to that order the naive answer agrees with the above exact answer.

Splitting $U(5) \rightarrow U(3) \times U(2)$

It is also possible to find the curve for $U(5)$ and from it to compute the $SU(5)$ answer by imposing the tracelessness constraint. However, the computation for $U(5)$ is more cumbersome than the $SU(5)$ counterpart. In this part of the section we will simply show the answer for the low energy effective superpotential and the computation can be found in Appendix A.

Since we now do not impose the tracelessness condition, λ is a free parameter, rather than a Lagrange multiplier. λ/g , m/g and Λ combine into a single expansion parameter

$$T^3 = \left(\frac{\Lambda}{\Delta} \right)^5,$$

with $\Delta = a_1 - a_2 = \sqrt{(\frac{m}{g})^2 - 4\frac{\lambda}{g}}$. The low energy superpotential is then given by,

$$W_{low} = 3W(a_1) + 2W(a_2) + g\Delta^3 (3T^2 + 2T^3 + 4T^4 + 10T^5 + \dots).$$

In the dual geometric picture we will see that $U(5)$ is the natural answer obtained, and then one has to impose the constraint to get the $SU(5)$ superpotential.

7. The analysis of the dual geometry

The dual geometry proposal gives rise to the superpotential of section 4.1:

$$-\frac{1}{2\pi i} W_{eff} = \sum_{i=1}^n N_i \Pi_i + \alpha \left(\sum_{i=1}^n S_i \right), \quad (7.1)$$

where Π_i 's are the periods of the dual cycles and the S_i 's are the sizes of the S^3 's as defined in (4.4) and (4.5).

Using (4.4) and (4.5), it is seen that under $\Lambda_0 \rightarrow e^{2\pi i} \Lambda_0$ the Π_i period will change by,

$$\Delta \Pi_i = -2 \left(\sum_{j=1}^n \pm S_j \right). \quad (7.2)$$

The factor of two comes from the fact that we are dealing with two copies of the x -plane connected by the n branch cuts. (See Figure 2) Let us choose the orientation of the fundamental periods to be clockwise, therefore, it is easy to see that we always get the upper sign in (7.2) for all i and j . We thus see that, in general, Π_i must depend on the cutoff Λ_0 as

$$\Pi_i = -\frac{2}{2\pi i} \left(\sum_{j=1}^n S_j \right) \log \Lambda_0 + \dots, \quad (7.3)$$

with \dots single valued under $\Lambda_0 \rightarrow e^{2\pi i} \Lambda_0$.

We now consider the full Λ_0 dependence. Consider the region of integration where x is large compared to all a_i 's. Therefore we can expand the effective one-form ω in x around $x = \infty$ and it is easy to see that,

$$\omega = \sqrt{W'(x)^2 + f_{n-1}} dx = \left(W'(x) + \frac{1}{2g_{n+1}} \frac{b_{n-1}}{x} + \mathcal{O}\left(\frac{1}{x^2}\right) \right) dx$$

where b_{n-1} is the coefficient of x^{n-1} in the deformation polynomial $f_{n-1}(x)$ and $W'(x) = g_{n+1} \prod_{j=1}^n (x - a_j)$. Integrating this we get,

$$\Pi_i = \dots + W(\Lambda_0) + \frac{b_{n-1}}{2g_{n+1}} \log \Lambda_0 + \mathcal{O}\left(\frac{1}{\Lambda_0}\right) \quad (7.4)$$

where \dots are the Λ_0 independent pieces. This allows us to make the following identification using (7.3) and (7.4).

$$b_{n-1} = -4g_{n+1} \sum_{j=1}^n S_j.$$

Comparing with (5.25), we see that we must have $\sum_j \langle S_j \rangle = -L\Lambda^N$, where both sides can be solved for in terms of the g_r and Λ . As mentioned in section 4.1, $W(\Lambda_0)$ is an irrelevant constant that can be ignored. However, we have to deal with the logarithmic dependence because we want to take $\Lambda_0 \rightarrow \infty$ at the end. Notice that, had we included deformations of degree higher than $n - 1$, more singular divergences would have appeared in (7.4) that

do not have a counterpart in the gauge theory side. This shows again that, as in (5.25), the deformation f in $F \sim W'^2 + f$ must have degree at most $n - 1$.

Since every Π_i has the same logarithmic divergence we can write the contribution to the superpotential as follows,

$$W_{eff} = \dots + 2\left(\sum_{i=1}^n N_i\right)\left(\sum_{j=1}^n S_j\right) \log \Lambda_0 - 2\pi i \alpha \left(\sum_{k=1}^n S_k\right)$$

Now it is clear that the only way to obtain finite expressions is to take α depending on Λ_0 such that

$$N \log \Lambda = N \log \Lambda_0 - \pi i \alpha \quad (7.5)$$

is finite. Using $\sum_{j=1}^n N_j = N$, we can replace Λ_0 in W_{eff} by the physical scale Λ of the $SU(N)$ theory.

Note that, for fixed Λ , the superpotential for a splitting of the form $KN \rightarrow \sum_{i=1}^n KN_i$ has a trivial K dependence:

$$-\frac{1}{2\pi i} W_{eff} = \sum_{i=1}^n KN_i \Pi_i = K \left(\sum_{i=1}^n N_i \Pi_i\right)$$

if we replace Λ_0 by Λ in the Π_i 's by using the α term. This matches with the results obtained from the gauge theory solution (5.30) using Chebyshev polynomials.

Some of the S_i dependence of Π_i can also be determined by using monodromy arguments. Consider the semiclassical regime, $|a_i^+ - a_i^-| \ll |a_j - a_k|$ for all i, j, k . Recall that $W'(x)^2 + f_{n-1}(x) = g_{n+1}^2 \prod_{k=1}^n (x - a_k^+)(x - a_k^-)$. In this regime S_i can be written as follows,

$$S_i = \frac{1}{2\pi i} W''(a_i) \int_{a_i^-}^{a_i^+} \sqrt{(x - a_i)^2 - \mu_{eff}} dx$$

where we have Taylor expanded $W'(x)^2 + f_{n-1}(x)$ around $x = a_i$ and

$$\mu_{eff} \equiv -\frac{1}{W''(a_i)^2} (f_{n-1}(a_i) + \dots).$$

Each S_i , in this limit, has been reduced to that of the single conifold, which has

$$S_i = W''(a_i) \mu_{eff}$$

up to a numerical coefficient. On the other hand, it is easy to see that under $\mu_{eff} \rightarrow e^{2\pi i} \mu_{eff}$, Π_i changes by $\Delta \Pi_i = S_i$. Therefore we conclude that,

$$\Pi_i = \frac{1}{2\pi i} S_i \log \mu_{eff} \dots = \frac{1}{2\pi i} S_i \log \frac{S_i}{W''(a_i)} + \dots$$

Finally, we want to consider what happens to Π_i when we move the j -th 3-sphere all the way around the i -th 3-sphere. This corresponds to changing $\Delta_{ij} = a_i - a_j$ to $e^{2\pi i} \Delta_{ij}$ leaving a_i fixed. Under this operation we get $\Delta \Pi_i = 2S_j$ (see Figure 3). Therefore,

$$\Pi_i = \dots + \frac{2}{2\pi i} \sum_{j \neq i} S_j \log \Delta_{ij}.$$

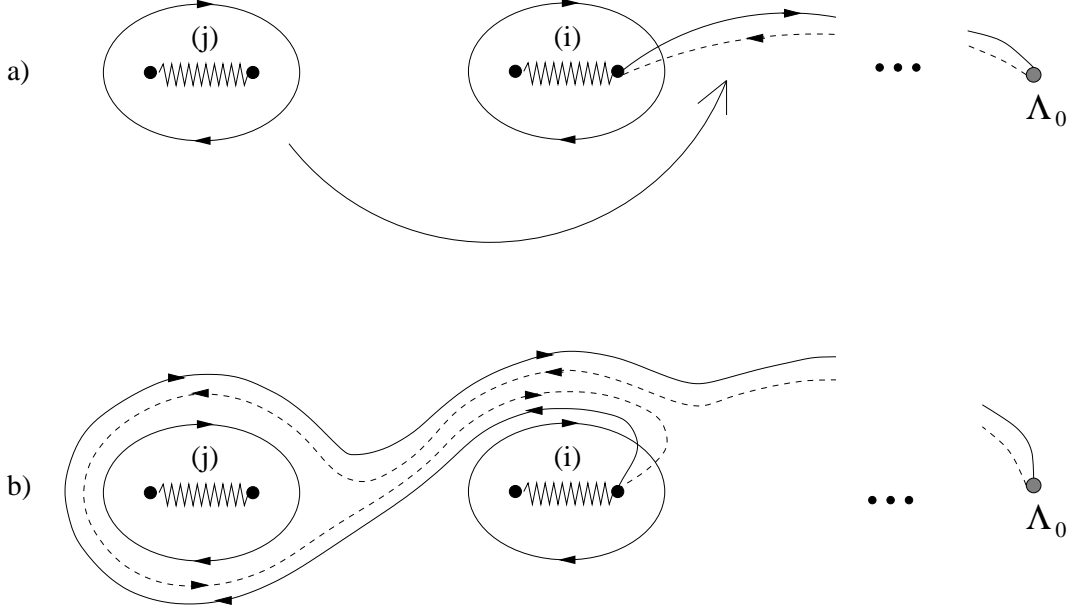


Figure 3: a) Contours of integration for S_j , S_i and Π_i before moving the j -th S^3 around the i -th S^3 . b) The Π_i contour goes around the j -th sphere after the operation in a).

Now we can collect all these partial results in order to write,

$$2\pi i \Pi_i = S_i \log \frac{S_i}{W''(a_i)} + 2 \sum_{j \neq i} S_j \log \Delta_{ij} - 2 \sum_{k=1}^n S_k \log \Lambda_0 + \dots$$

Plugging this back in (7.1) and collecting all the S_i pieces, we get

$$W_{eff} = \sum_{i=1}^n S_i \log \left(\frac{W''(a_i)^{N_i} \prod_{j \neq i} \Delta_{ij}^{-2N_j} \Lambda^{2N}}{S_i^{N_i}} \right) + \dots,$$

with the \dots single valued.

Comparing this to (5.8) and (5.6) we see that we have re-derived the approximate W_{eff} obtained in section 5.1 as well as the naive threshold matching relations. However, the above analysis can not rule out further corrections to each Π_i and hence to W_{eff} in the form of a power series in S_i 's. Indeed, as we will discuss in detail for the case of the cubic superpotential, there is generally an infinite power series in S_i 's which corrects the above expression.

8. Cubic superpotential from geometry: An explicit computation

In this section we consider the $n = 2$ case, deforming the $\mathcal{N} = 2$ theory by $W_{tree} = \lambda u_1 + m u_2 + g u_3$. This was discussed in detail from the gauge theory perspective in section 6. We now focus on the geometry side of the duality. In order to get the contribution of the fluxes to the superpotential, we need to compute the periods of the relevant cycles in the geometry. For this $n = 2$ case, (7.1) gives

$$-\frac{1}{2\pi i} W_{eff} = N_1 \Pi_1 + N_2 \Pi_2 + \alpha(S_1 + S_2). \quad (8.1)$$

The fundamental periods are given as in (4.4) by,

$$S_1 = \frac{1}{2\pi i} \int_{x_3}^{x_4} \omega \quad S_2 = \frac{1}{2\pi i} \int_{x_1}^{x_2} \omega \quad (8.2)$$

and the dual periods by

$$\Pi_1 = \frac{1}{2\pi i} \int_{x_3}^{\Lambda_0} \omega \quad \Pi_2 = \frac{1}{2\pi i} \int_{-\Lambda_0}^{x_1} \omega \quad (8.3)$$

where we have denoted by x_i the roots of the quartic polynomial $W'(x)^2 + f_1(x)$ appearing in the definition of the effective one-form instead of a_i^+, a_i^- as in last section, in order to simplify the notation.

To compute the effective superpotential, we need to express the dual periods Π_1 and Π_2 in terms of the fundamental periods S_1 and S_2 . Since, on the gauge theory side, one does not have the exact answer for the superpotential in terms of the glueball fields, we need to integrate out the S_i , fixing them at their supersymmetric vacua $\langle S_i \rangle$. This will give $W_{exact}(\lambda, m, g, \Lambda)$, which can be compared with the gauge theory results.

Recall that λ is a free parameter only for the $U(N)$ theory. For $SU(N)$, which we will also compare, λ is a Lagrange multiplier imposing (quantum) tracelessness; this will fix λ in terms of m, g and Λ and the N_i .

8.1. Computation of the periods

As discussed in the general case in section 7, only by using monodromy arguments it is possible to show the general form of the S_i dependence of the dual periods. In our case, this reads,

$$\Pi_1 = \frac{1}{2\pi i} \left(W(\Lambda_0) - W(a_1) + S_1 \log \frac{S_1}{g\Delta} - S_1 + 2S_2 \log \Delta - 2(S_1 + S_2) \log \Lambda_0 + P \right) \quad (8.4)$$

where $P = P(S_1, S_2)$ is an infinite power series in S_1 and S_2 , $\Delta = a_1 - a_2$ and $W(x) = (1/3)gx^3 + (1/2)mx^2 + \lambda x$. Recall that $W'(x) = g(x - a_1)(x - a_2)$ was introduced in section 6. Use has also been made of $W''(a_1) = g\Delta$.

The explicit computation of $P(S_1, S_2)$ can be found in Appendix B up to order S_i^4 where a method to compute higher order contributions is also given. Here we will only show the result for Π_1 and Π_2 that will be used later in this section.

$$\begin{aligned} 2\pi i \Pi_1 = & W(\Lambda_0) - W(a_1) + S_1(\log \frac{S_1}{g\Delta} - 1) + 2S_2 \log \Delta - 2(S_1 + S_2) \log \Lambda_0 + \\ & + g(\Delta)^3 \left[\frac{1}{(g\Delta^3)^2} (2S_1^2 - 10S_1S_2 + 5S_2^2) + \frac{1}{(g\Delta^3)^3} \left(\frac{32}{3}S_1^3 - 91S_1^2S_2 + \right. \right. \\ & + 118S_1S_2^2 - \frac{91}{3}S_2^3) + \frac{1}{(g\Delta^3)^4} \left(\frac{280}{3}S_1^4 - \frac{3484}{3}S_1^3S_2 + 2636S_1^2S_2^2 + \right. \\ & \left. \left. - \frac{5272}{3}S_1S_2^3 + \frac{871}{3}S_2^4 \right) + \mathcal{O}\left(\frac{S^5}{(g\Delta^3)^5}\right) \right] \end{aligned}$$

and,

$$\begin{aligned} 2\pi i \Pi_2 = & W(-\Lambda_0) - W(a_2) + S_2(\log \frac{S_2}{g\Delta} - 1) + 2S_1 \log \Delta - 2(S_1 + S_2) \log \Lambda_0 + \\ & - g(\Delta)^3 \left[\frac{1}{(g\Delta^3)^2} (2S_2^2 - 10S_1S_2 + 5S_1^2) - \frac{1}{(g\Delta^3)^3} \left(\frac{32}{3}S_2^3 - 91S_2^2S_1 + \right. \right. \\ & + 118S_2S_1^2 - \frac{91}{3}S_1^3) + \frac{1}{(g\Delta^3)^4} \left(\frac{280}{3}S_2^4 - \frac{3484}{3}S_2^3S_1 + 2636S_2^2S_1^2 + \right. \\ & \left. \left. - \frac{5272}{3}S_2S_1^3 + \frac{871}{3}S_1^4 \right) + \mathcal{O}\left(\frac{S^5}{(g\Delta^3)^5}\right) \right] \end{aligned}$$

8.2. Low Energy Superpotential

In order to compute the low energy superpotential we have to integrate out S_1 and S_2 from the effective superpotential. In order to do this in practice, it is convenient to define

$$x \equiv \frac{S_1}{g\Delta^3} \quad y \equiv \frac{S_2}{(-g\Delta^3)}$$

In term of these new variable the dual periods can be written as follows,

$$\begin{aligned} \Pi_1(x, y) &= \frac{1}{2\pi i} (-W(a_1) + (g\Delta^3)\mathcal{F}(x, y)) \\ \Pi_2(x, y) &= \frac{1}{2\pi i} (-W(a_2) + (-g\Delta^3)\mathcal{F}(y, x)) \end{aligned}$$

where,

$$\begin{aligned}\mathcal{F}(x, y) = & x(\log x - 1) - 2(x - y) \log\left(\frac{\Lambda_0}{\Delta}\right) + (2x^2 + 10xy + 5y^2) + \left(\frac{32}{3}x^3 + 91x^2y + \right. \\ & \left. 118xy^2 + \frac{91}{3}y^3\right) + \left(\frac{280}{3}x^4 + \frac{3484}{3}x^3y + 2636x^2y^2 + \frac{5272}{3}xy^3 + \frac{871}{3}y^4\right) + \dots\end{aligned}$$

Note that we have removed the irrelevant constants $W(\Lambda_0)$ in Π_1 and $W(-\Lambda_0)$ in Π_2 . Now the effective superpotential is given by,

$$-\frac{1}{2\pi i}W_{eff}(x, y) = N_1\Pi_1 + N_2\Pi_2 + \alpha g\Delta^3(x - y) \quad (8.5)$$

Let us separate the contributions to (8.5) as,

$$W_{eff}(x, y) = W_{cl} + W_{np}(x, y)$$

where $W_{cl} = N_1W(a_1) + N_2W(a_2)$ and $W_{np}(x, y) = g(\Delta^3)(-N_1\mathcal{F}(x, y) + N_2\mathcal{F}(y, x))$. In this expression, the cut off Λ_0 gets combined with the bare coupling α to generate what we identify with the gauge theory scale of the underlying $N = 2$ $SU(N)$ Yang-Mills theory Λ as in (7.5).

Having identified the gauge theory scale Λ we can proceed to integrate out S_1 and S_2 or equivalently x and y . The equations that need to be solved are,

$$\frac{\partial W_{eff}}{\partial x} = 0 \quad \frac{\partial W_{eff}}{\partial y} = 0$$

The leading order can be easily extracted and reads,

$$N_1 \log(x) = 2(N_1 + N_2) \log\left(\frac{\Lambda}{\Delta}\right) \quad N_2 \log(y) = 2(N_1 + N_2) \log\left(\frac{\Lambda}{\Delta}\right)$$

Now we can see the appearance of the N_1N_2 vacua of the gauge theory from the solutions to the above equations, namely,

$$x^{N_1} = \left(\frac{\Lambda}{\Delta}\right)^{2(N_1+N_2)} \quad y^{N_2} = \left(\frac{\Lambda}{\Delta}\right)^{2(N_1+N_2)}$$

It is useful to define the expansion parameter

$$T \equiv \left(\frac{\Lambda}{\Delta}\right)^{\frac{2(N_1+N_2)}{N_1N_2}}$$

and the solution is then given by

$$x = T^{N_2}, \quad y = T^{N_1}$$

where the choice of the N_1N_2 -th root will determine the vacuum.

Note that the meaning of leading order depends on the values of N_1 and N_2 . Assuming a power series expansion for x and y in T we can compute order by order W_{low} . This gives us the answer for the $U(N)$ theory. To obtain the answer for $SU(N)$, we only have to impose that the quantum trace of the chiral superfield be zero: $\langle Tr\Phi \rangle = \frac{\partial W_{low}(\lambda)}{\partial \lambda} = 0$. This should be imposed order by order in T .

8.3. Quantum tracelessness

Let us start by writing,

$$W_{low}(\lambda, \Lambda) = N_1 W(a_1) + N_2 W(a_2) + g \Delta^3 P(T)$$

with $P(T) = -N_1 \mathcal{F}(x_{(T)}, y_{(T)}) + N_2 \mathcal{F}(y_{(T)}, x_{(T)})$. It is then easy to see that

$$\frac{\partial W_{low}(\lambda, \Lambda)}{\partial \lambda} = N_1 a_1 + N_2 a_2 - 2\Delta \left(3P(T) - \frac{2(N_1 + N_2)}{N_1 N_2} T \frac{dP(T)}{dT} \right) \quad (8.6)$$

where it was important to remember that T itself depends on λ through Δ . Therefore we are forced to define a better expansion parameter given by,

$$t = \left(\frac{\Lambda}{\Delta_c} \right)^{\frac{2(N_1 + N_2)}{N_1 N_2}}$$

where Δ_c is computed using the Lagrange multiplier obtained by solving the classical tracelessness constraint,

$$\frac{\lambda_c}{g} = -\frac{N_1 N_2}{(N_1 - N_2)^2} \left(\frac{m}{g} \right)^2 \quad (8.7)$$

Having found $\lambda = \lambda(t)$ such that the quantum trace (8.6) vanishes, we can use it to compute the low energy superpotential for our $SU(N)$ theory that is given now as a power expansion in t . It is possible to give an explicit formula for the first two terms, i.e, the classical contribution and the first quantum correction for any N_1 and N_2 . Higher order corrections have to be computed independently in each case. Assuming that $N_2 < N_1$, we get,

$$W_{low}(t) = \frac{1}{6} \frac{m^3}{g^2} \frac{N_1 N_2 (N_1 + N_2)}{(N_1 - N_2)^2} \left[1 + \frac{6(N_1 + N_2)^2}{N_2 (N_1 - N_2)} t^{N_2} + \mathcal{O}(t^{N_2+1}) \right].$$

8.4. Examples

Let us consider the different cases for which the deformed $\mathcal{N} = 2$ field theory results have been computed in section 6, in order to compare the answer with that of the geometry.

$$U(3N) \rightarrow U(2N) \times U(N)$$

We only need to consider the case $U(3) \rightarrow U(2) \times U(1)$. As we saw in section 6.1 this is particularly simple from the field theory perspective, where $W_{eff} = W_{cl} \pm 2g\Lambda^3$, with only

one quantum correction term. In order to reproduce this simple result, some miraculous cancellations have to occur order by order in our series. Since we have computed the dual periods up to order S_1^4 and therefore the effective superpotential up to order $x^4 \sim t^4$, we can not compare orders equal or higher than t^5 even though they already appear in our computation in the form xy^2 or x^3y since $y \sim t^2$.

Let $N_1 = 2$ and $N_2 = 1$. Integrating out x and y we get,

$$x_{(T)} = T(1 + T + 10T^2 + 140T^3 + \dots) \quad y_{(T)} = T^2(1 + 10T + 140T^2 + \dots)$$

Plugging this back in W_{eff} we get the answer for the $U(3)$ case,

$$W_{low}(T) = W_{cl} + g\Delta^3(2T + \mathcal{O}(T^5))$$

which is consistent with the exact answer $W = W_{cl} + 2g\Delta^3T$ discussed in section 6.1, to the order we have computed. One might worry that imposing quantum tracelessness for $SU(3) \rightarrow SU(2) \times U(1)$ could result in T being a complicated expansion in terms of t . However, one can check that the classical trace is not corrected quantum mechanically in this case and therefore $T = t$. We thus have

$$W_{low}(t) = \frac{m^3}{g^2}(1 + 54t + \mathcal{O}(t^5)),$$

and, recalling the definition of $t = \pm \left(\frac{g\Lambda}{3m}\right)^3$, we get

$$W_{low}(t) = \frac{m^3}{g^2} \pm 2g\Lambda^3,$$

in perfect agreement with the field theory result.

We can also use the geometry analysis to obtain the gauge coupling of the IR $U(1)$ gauge theory photon, and compare with the field theory analysis. The field theory result obtained in section 6.1 is that the original $SU(3)$ curve degenerates as $P_3^2 - 4\Lambda^6 = (x - m/g)^2 F_4(x)$, with $g^2 F_4(x) = W'(x)^2 \mp 4g^2 \Lambda^3(x + 2m/g)$. The remaining massless photon has gauge coupling given by the complex modulus τ of the torus $y^2 = F_4(x)$. This matches perfectly with the geometry result if, at the extremum of our effective superpotential for S , we have $f_1(x; \langle S \rangle) = \mp 4g^2 \Lambda^3(x + 2m/g)$. Strikingly, this is indeed the case.

$$U(5N) \rightarrow U(3N) \times U(2N)$$

In this case the deformed $\mathcal{N} = 2$ field theory analysis predicts an infinite series discussed in section 6.1. From the dual geometry, to the order we have computed, we will

be able to compare up to order t^9 because $x \sim t^2$ and $y \sim t^3$, therefore the t^{10} receives corrections from the x^5 .

Let $N_1 = 3$ and $N_2 = 2$. Integrating out x and y we get,

$$x(T) = T^2 \left(1 + \frac{8}{3}T^2 - \frac{10}{3}T^3 + a_4T^4 + a_5T^5 + a_6T^6 + a_7T^7 + \dots \right)$$

and

$$y(T) = T^3(1 + 5T^2 + 11T^3 + b_4T^4 + b_5T^5 + b_6T^6 + \dots)$$

The undetermined coefficients are shown to stress the fact that they do not contribute to the order we are computing, despite being allowed by naive power counting. Plugging this back in W_{eff} we get the answer for $U(5)$,

$$W_{low}(T) = W_{cl} + g\Delta^3 \left(3T^2 - 2T^3 + 4T^4 - 10T^5 + \frac{85}{3}T^6 - \frac{266}{3}T^7 + \frac{8170}{27}T^8 + \right. \\ \left. - \frac{3332}{3}T^9 + \dots \right).$$

In this case, we do have to take care with the quantum corrections to the trace, in order to get the correct $SU(5)$ superpotential. It turns out that

$$\frac{\lambda}{g} = \frac{\lambda_c}{g} \left(1 - \frac{25}{3}t^2 + \frac{100}{3}t^3 - \frac{550}{3}t^4 + \frac{10400}{9}t^5 - 7875t^6 + \frac{508300}{9}t^7 - \frac{11338250}{27}t^8 + \dots \right).$$

Using this to compute $T = T(t)$, $a_1 = a_1(t)$, and $a_2 = a_2(t)$, and plugging back in the effective superpotential, we get

$$W_{low}(t) = \frac{250}{2} \frac{m^3}{g^2} \left(\frac{1}{25} + 3t^2 - 2t^3 + 6t^4 - 26t^5 + 135t^6 - 782t^7 + \frac{14630}{3}t^8 + \right. \\ \left. - 32076t^9 + \dots \right).$$

This is in perfect agreement with the deformed $\mathcal{N} = 2$ field theory answer.

$$U(5N) \rightarrow U(4N) \times U(N)$$

The deformed $\mathcal{N} = 2$ field theory analysis again predicts an infinite series for W_{eff} . Again, this is also seen from the geometry dual, and we will be able to compare up to

order t^4 since we have computed the dual periods to order S_1^4 . Let $N_1 = 4$ and $N_2 = 1$. Integrating out x and y we get,

$$x(T) = T \left(1 - \frac{3}{2}T - \frac{47}{8}T^2 - \frac{73}{2}T^3 + \dots \right), \quad y(T) = T^4 + \mathcal{O}(T^5).$$

Plugging this back in the effective superpotential we get the $U(4)$ answer,

$$W_{low} = W_{cl} + g\Delta^3(4T - 3T^2 - \frac{47}{6}T^3 - \frac{75}{2}T^4 + \dots).$$

For the $SU(4)$ case, the vanishing of the quantum corrected trace implies that,

$$\frac{\lambda}{g} = \frac{\lambda_c}{g} \left(1 + \frac{25}{3}t + \frac{25}{9}t^2 + \frac{175}{72}t^3 + \dots \right).$$

Using this as in the previous case, we finally get the low energy superpotential to be

$$W_{low} = \frac{125}{27} \frac{m^3}{g^2} \left(\frac{2}{25} + 4t - \frac{1}{3}t^2 - \frac{7}{54}t^3 - \frac{5}{54}t^4 + \mathcal{O}(t^5) \right).$$

This exactly agrees, to this order, with the expected answer.

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Appendix A. Deformed $\mathcal{N} = 2$ field theory analysis for $U(5) \rightarrow U(3) \times U(2)$

We here find the supersymmetric vacua of the deformed $\mathcal{N} = 2$ theory for one of the splittings of $U(5)$. The analysis goes along much the same lines as for $SU(5)$. We parameterize

$$P_5(x) = (x - (q + a))^2(x - (q + b))^2(x - (q + c)) \mp 1. \quad (\text{A.1})$$

For $SU(5)$, q was fixed by the tracelessness condition but now it is a free parameter. Since a and b appear in a symmetric way it turns out to be useful to define $s = a + b + 2q$ and $k = (a + q)(b + q)$. The constraints are now given by,

$$k = q^2 - q(2q - s) + 2c(2q - s) \quad 4(2q - s)^2 c^3 = \pm 1$$

From (A.1) we can read off u_1 , u_2 , and u_3 using that,

$$P_5(x) = x^5 - u_1 x^4 + \left(\frac{1}{2}u_1^2 - u_2\right)x^3 + \left(\frac{1}{6}u_1^3 + u_1 u_2 - u_3\right)x^2 + \dots$$

Plugging $u_i = u_i(q, s, c)$ in W_{eff} and introducing a Lagrange multiplier in order to impose the constraint left after we eliminate k , we get,

$$W_{eff} = gu_3 + mu_2 + \lambda u_1 + h(\mp \Lambda^5 - 4(2q - s)^2 c^3)$$

The equations we need to solve are given by $\partial W_{eff}/\partial c = 0$, $\partial W_{eff}/\partial s = 0$, and $\partial W_{eff}/\partial q = 0$. Using the first to eliminate h in the second and the third, these equations simplify to,

$$\lambda + m(-q + c + s) + g(q^2 + c^2 + 3cs + s^2 - 2q(2c + s)) = 0$$

$$-5\lambda - m(q + c + 2s) - g(5q^2 - 14qc + c^2 - 4qs + 8cs + 2s^2) = 0$$

$$4(2q - s)^2 c^3 = \pm \Lambda^5$$

In order to find an expansion parameter around the classical solution we have to take the limit $\Lambda = 0$ and solve the equations. We find that,

$$q = \frac{-m + \sqrt{m^2 - 4g\lambda}}{2g} \quad s = -\frac{m}{g}$$

Therefore, $(2q - s) = \sqrt{\frac{m^2}{g^2} - 4\frac{\lambda}{g}} = \Delta$ and it is clear that the expansion parameter is T given by $T^6 = (\frac{\Lambda}{\Delta})^{10}$. Again, there are six possible solutions giving the six possible vacua $N_1 N_2 = 6$, since $N_1 = 2$ and $N_2 = 3$.

Solving these equations assuming a power expansion in T for $s = s(T)$, $q = q(T)$ and $c = c(T)$, we get after plugging back in the effective superpotential,

$$W_{low} = 3W(a_1) + 2W(a_2) + g\Delta (3T^2 + 2T^3 + 4T^4 + 10T^5 + \dots)$$

where $W(x) = \frac{g}{3}x^3 + \frac{m}{2}x^2 + \lambda x$, $W'(x) = g(x - a_1)(x - a_2)$ and $\Delta = a_1 - a_2$.

Appendix B. Computation of Periods for the cubic superpotential

In this appendix we will show the explicit computation of the corrections $P(S_1, S_2)$ in the expression for Π_1 in (8.4).

The computation of $P(S_1, S_2)$ will not be done directly in terms of S_1 and S_2 , we will write all four periods in terms of two new variables Δ_{21} and Δ_{43} - to be defined below - and at the end we will recollect $P(S_1, S_2)$. This procedure can be done systematically up to any order in S_i 's.

Computation:

For practical purposes we will write the effective one-form as follows

$$dx \sqrt{W'^2(x) + f_1(x)} = dx \, g \sqrt{(x - x_1)(x - x_2)(x - x_3)(x - x_4)} \quad (\text{B.1})$$

It is also convenient to define new variables given by,

$$\begin{aligned} \Delta_{21} &\equiv \frac{1}{2}(x_2 - x_1) & \Delta_{43} &\equiv \frac{1}{2}(x_4 - x_3) \\ Q &\equiv \frac{1}{2}(x_1 + x_2 + x_3 + x_4) & I &\equiv \frac{1}{2}((x_3 + x_4) - (x_1 + x_2)) \end{aligned}$$

It is clear that since $f_1(x)$ is considered a small perturbation we will have

$$|\Delta_{21}| \sim |\Delta_{43}| \ll |I|.$$

We will use this in order to expand all four periods in powers of Δ_{21} and Δ_{43} .

Let us consider S_1 . For this we change variables to $y = x - \frac{1}{2}(x_1 + x_2)$ and the integral becomes:

$$S_1 = \frac{g}{2\pi} \int_{y_3}^{y_4} \sqrt{(y - y_3)(y - y_4)} \sqrt{y^2 - \Delta_{21}^2}$$

Expanding the second square root for Δ_{21} small, each term in the series can be computed explicitly and it is most easily given in terms of a generating function,

$$F(a) \equiv -\pi \sqrt{(y_3 + a)(y_4 + a)} + \frac{\pi}{2}(y_3 + y_4 + 2a) \quad (\text{B.2})$$

as follows,

$$S_1 = \frac{g}{32}(y_3 + y_4)(y_4 - y_3)^2 + \frac{g}{2\pi} \sum_{n=1}^{\infty} c_n \Delta_{21}^{2n} F^{(n)}(0)$$

where c_n are the coefficients in the expansion of $\sqrt{1 - x}$ and $F^{(n)}(a)$ is the n -th derivative with respect to a .

The explicit answer has the following structure,

$$S_1 = \frac{g}{4}\Delta_{43}^2 I - \frac{g}{2I}K(\Delta_{21}^2, \Delta_{43}^2, I^2) \quad (\text{B.3})$$

where

$$K(x, y, z) = \frac{1}{4}xy \left(1 + \frac{1}{4z}(x+y) + \frac{1}{8z^2}(x+y)^2 + \frac{1}{8z^2}xy + \dots \right)$$

It is important to notice that this is symmetric in (x, y) , namely, $K(x, y, z) = K(y, x, z)$. This allows us to write,

$$S_2 = -\frac{g}{4}\Delta_{21}^2 I + \frac{g}{2I}K(\Delta_{21}^2, \Delta_{43}^2, I^2) \quad (\text{B.4})$$

Let us now compute the dual periods starting with Π_1 . In this case we can use the same expansion as before for S_1 , however, we have to keep in mind that Λ_0 will be taken to infinity at the end and therefore we shall discard any contribution of order Λ_0^{-1} or higher in an expansion around infinity.

In this case it is also useful to define a generating function,

$$G(a) = \sqrt{(I+a)^2 - \Delta_{43}^2} \log \left(\frac{\sqrt{(I+a) + \Delta_{43}} + \sqrt{(I+a) - \Delta_{43}}}{\sqrt{(I+a) + \Delta_{43}} - \sqrt{(I+a) - \Delta_{43}}} \right) \quad (\text{B.5})$$

and the answer is given by,

$$\begin{aligned} \frac{2\pi i}{g}\Pi_1 = & \frac{1}{3}\Lambda_0^3 - \frac{1}{2}Q\Lambda_0^2 + \frac{1}{4}(Q^2 - I^2 - 2(\Delta_{43}^2 + \Delta_{21}^2))\Lambda_0 + \frac{1}{2}I(\Delta_{21}^2 - \Delta_{43}^2) \log \Lambda_0 \\ & - \frac{1}{24}(I+Q)^3 + \frac{1}{8}I(I+Q)^2 + \frac{1}{4}\Delta_{21}^2(I+Q) + \frac{1}{4}\Delta_{43}^2 Q + \\ & \frac{1}{2}I(\Delta_{43}^2 - \Delta_{21}^2) \log(2\Delta_{43}) + \sum_{n=1}^{\infty} c_n \Delta_{21}^{2n} G^{(n)}(0) \end{aligned} \quad (\text{B.6})$$

where c_n are as before the coefficients of the power expansion of $\sqrt{1-x}$.

This result is not enough because we want it to show only explicit dependence on the classical superpotential parameters m, g, λ and the two deformation parameters Δ_{21} and Δ_{43} . In order to do this we only have to realize that since $f_1(x)$ in (B.1) is of degree one and $W'^2(x)$ of degree four, then the coefficients of x^3 and x^2 are given in terms of the classical roots a_1 and a_2 . This allows us to write,

$$Q = a_1 + a_2 \quad I^2 = (a_1 - a_2)^2 - 2(\Delta_{21}^2 + \Delta_{43}^2)$$

Using this, (B.3) and (B.4) we can explicitly compare order by order in Δ_{21} and Δ_{43} the two expressions for Π_1 given by (B.6) and (8.4) to obtain the following result,

$$\begin{aligned}
2\pi i \Pi_1 = & W(\Lambda_0) - W(a_1) + S_1 \left(\log \frac{S_1}{g\Delta} - 1 \right) + 2S_2 \log \Delta - 2(S_1 + S_2) \log \Lambda_0 + \\
& + g(\Delta)^3 \left[\frac{1}{(g\Delta^3)^2} (2S_1^2 - 10S_1S_2 + 5S_2^2) + \frac{1}{(g\Delta^3)^3} \left(\frac{32}{3}S_1^3 - 91S_1^2S_2 + \right. \right. \\
& + 118S_1S_2^2 - \frac{91}{3}S_2^3 \Big) + \frac{1}{(g\Delta^3)^4} \left(\frac{280}{3}S_1^4 - \frac{3484}{3}S_1^3S_2 + 2636S_1^2S_2^2 + \right. \\
& \left. \left. - \frac{5272}{3}S_1S_2^3 + \frac{871}{3}S_2^4 \right) + \mathcal{O} \left(\frac{S^5}{(g\Delta^3)^5} \right) \right].
\end{aligned}$$

Likewise we can get Π_2 from the above result by simply exchanging $a_1 \leftrightarrow a_2$, $S_1 \leftrightarrow S_2$, $\Delta \leftrightarrow -\Delta$ and $\Lambda_0 \leftrightarrow -\Lambda_0$. This leads to,

$$\begin{aligned}
2\pi i \Pi_2 = & W(-\Lambda_0) - W(a_2) + S_2 \left(\log \frac{S_2}{g\Delta} - 1 \right) + 2S_1 \log \Delta - 2(S_1 + S_2) \log \Lambda_0 + \\
& - g(\Delta)^3 \left[\frac{1}{(g\Delta^3)^2} (2S_2^2 - 10S_1S_2 + 5S_1^2) - \frac{1}{(g\Delta^3)^3} \left(\frac{32}{3}S_2^3 - 91S_2^2S_1 + \right. \right. \\
& + 118S_2S_1^2 - \frac{91}{3}S_1^3 \Big) + \frac{1}{(g\Delta^3)^4} \left(\frac{280}{3}S_2^4 - \frac{3484}{3}S_2^3S_1 + 2636S_2^2S_1^2 + \right. \\
& \left. \left. - \frac{5272}{3}S_2S_1^3 + \frac{871}{3}S_1^4 \right) + \mathcal{O} \left(\frac{S^5}{(g\Delta^3)^5} \right) \right]
\end{aligned}$$

This completes our computation of the periods.

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